# THE GANDY-HYLAND FUNCTIONAL AND A HITHERTO UNKNOWN COMPUTATIONAL ASPECT OF NONSTANDARD ANALYSIS

SAM SANDERS

ABSTRACT. In this paper, we highlight a new *computational* aspect of Nonstandard Analysis relating to higher-order recursion theory. In particular, we prove that the *Gandy-Hyland functional* equals a primitive recursive functional *involving nonstandard numbers* inside Nelson's *internal set theory*. From this classical and ineffective proof in Nonstandard Analysis, a term from Gödel's system $\mathsf{T}$ is extracted which computes the Gandy-Hyland functional in terms of a *modulus-of-continuity functional* and a special case of the *fan functional*. We obtain several similar relative computability results *not involving Nonstandard Analysis* from their associated nonstandard theorems, in particular involving the *weak continuity functional*. By way of reversal, we show that certain relative computability results, called *Herbrandisations*, also imply the nonstandard theorem from whence they were obtained. Thus, we establish a direct two-way connection between the field *Computability* (in particular theoretical computer science) and the field *Nonstandard Analysis*.

## 1. INTRODUCTION

The aim of this paper is to highlight a new *computational* aspect of Nonstandard Analysis relating to higher-order computability theory. Our object of study is the *Gandy-Hyland functional*, which was introduced in [12] as an example of a higher-type functional not computable (in the sense of Kleene's S1-S9 from [17, Def. 1.10]) in the *fan functional* over the total continuous functionals (See [17, 4.61] and Section 3.1.1). The Gandy-Hyland functional $\Gamma$ is defined as follows:

$$(\exists\Gamma^3)(\forall Y^2 \in C, s^0)\big[\Gamma(Y^2, s^0) = Y\big(s * 0 * (\lambda n^0)\Gamma(Y, s*(n+1))\big)\big], \tag{$\mathsf{GH}$}$$

where '$Y^2 \in C$' is the usual definition of (pointwise) continuity on Baire space as in (4.1). Additional notations are introduced in Section 2.3.

The definition ($\mathsf{GH}$) apparently exhibits non-well-founded self-reference: Indeed, in order to compute $\Gamma$ at $s^0$, one needs the values of $\Gamma$ at all child nodes of $s^0$, as is clear from the right-hand side of ($\mathsf{GH}$). In turn, to compute the value of $\Gamma$ at the child nodes of $s$, one needs the value of $\Gamma$ at all grand-child nodes of $s$, and so on. Hence, repeatedly applying the definition of $\Gamma$ seems to result in a non-terminating recursion. By contrast, *primitive recursion* is well-founded as it reduces the case for $n+1$ to the case for $n$, and the case for $n = 0$ is given.

DEPARTMENT OF MATHEMATICS, GHENT UNIVERSITY, BELGIUM & MUNICH CENTER FOR MATHEMATICAL PHILOSOPHY, LMU MUNICH, GERMANY

*E-mail address*: sasander@me.com.

In Section 4.1, we show that inside Nelson's *internal set theory* (See [16] and Section 2), the following *primitive recursive* (by [9, Theorem 18]) functional

$$H(Y^2, s^0, M) = \begin{cases} Y(\overline{s}M * 00\dots) & |s| \geq M \\ Y(s * 0 * H(Y, s*1, M) * \dots * H(Y, s*M, M) * 00\dots) & \text{otherwise} \end{cases}$$

equals the $\Gamma$-functional from (GH) for standard input and any nonstandard number $M^0$. Note that one need only apply the definition of $H$ at most $M$ times to terminate in its first case. In other words, the extra case '$|s| \geq M$' provides a nonstandard stopping condition which 'unwinds' the non-terminating recursion in $\Gamma$ to the terminating one in $H$. Or: one can trade in self-reference for nonstandard numbers. Thus, we shall refer to $H$ as the *canonical approximation*[1] of $\Gamma$.

We work in P, a fragment of Nelson's *internal set theory* based on Gödel's T, both introduced in Section 2. The proof in Section 4 that $H(\cdot, M)$ and $\Gamma(\cdot)$ are equal for standard inputs and nonstandard $M^0$, takes place in P augmented with a nonstandard continuity axiom NPC and a nonstandard bar induction axiom STP. This is a natural setting for $\Gamma$, as it is modified bar recursion in disguise ([6, §4]).

From the aforementioned proof in P regarding $\Gamma$ and $H$, we shall extract a term $t$ from Gödel's T and a proof in higher-order Peano arithmetic that $t$ computes the Gandy-Hyland functional in terms of a special case of the fan functional (originating from the bar induction axiom) and a modulus-of-continuity functional (originating from the nonstandard continuity axiom). Conceptually, it is important to note that this final proof, as well as the term $t$, *does not involve Nonstandard Analysis*, and that the extraction of the term $t$ from the proof proceeds via an algorithm. In Sections 4.3 to 4.6, we obtain further nonstandard results from which we extract related relative computability results.

Furthermore, it is a natural 'Reverse Mathematics[2] style' question if from a relative computability result (obtained from Nonstandard Analysis), the 'original' nonstandard theorem can be re-obtained. In answer to this question, we show in Section 4.2 that (a proof of) the original nonstandard theorem (that the Gandy-Hyland functional $\Gamma(\cdot)$ equals $H(\cdot, M)$ for all standard inputs and nonstandard $M$) follows from (a proof of) a certain natural relative computability result, called the *Herbrandisation* of the original nonstandard theorem.

In conclusion, while these relative computability results are not necessarily deep or surprising in and off themselves, *the methodology by which we arrive at them* constitutes the real surprise of this paper, namely *a new computational aspect of Nonstandard Analysis*: From a classical-logic proof in which no attention to computability is given at all, and in which Nonstandard Analysis is freely used, we obtain a relative computability result in a straightforward way. With some attention to detail, a natural relative computability result, called the Herbrandisation, allows us to re-obtain the original nonstandard theorem. In this way, we establish a direct two-way connection between the field Computability (in particular theoretical computer science) and the field Nonstandard Analysis.

[1]As it turns out, the Gandy-Hyland functional may also be viewed as the limit for $n \to \infty$ of a certain operator $\Delta_\Gamma^n$ ([15, 8.3.10]). While the nonstandard approach from this paper certainly applies to this limit result, it is our opinion that the canonical approximation via $H$ is simpler.

[2]For an introduction to the foundational program *Reverse Mathematics*, we refer the reader to Simpson's monograph [21].

## 2. About and around internal set theory

In this section, we introduce the base theory $\mathsf{P}$ in which we will work. In two words, $\mathsf{P}$ is a conservative extension of Gödel's system $\mathsf{T}$ with certain axioms from Nelson's *Internal Set Theory* ([16]) based on the approach from [4, 5].

### 2.1. Internal set theory and its fragments.

In this section, we discuss Nelson's *internal set theory*, first introduced in [16], and its fragments from [4].

In Nelson's *syntactic* approach to Nonstandard Analysis ([16]), as opposed to Robinson's semantic one ([18]), a new predicate 'st($x$)', read as '$x$ is standard' is added to the language of $\mathsf{ZFC}$, the usual foundation of mathematics. The notations $(\forall^{\text{st}}x)$ and $(\exists^{\text{st}}y)$ are short for $(\forall x)(\text{st}(x) \to \dots)$ and $(\exists y)(\text{st}(y) \wedge \dots)$. A formula is called *internal* if it does not involve 'st', and *external* otherwise. The three external axioms *Idealisation*, *Standard Part*, and *Transfer* govern the new predicate 'st'; They are respectively defined[3] as:

- (I) $(\forall^{\text{st fin}}x)(\exists y)(\forall z \in x)\varphi(z,y) \to (\exists y)(\forall^{\text{st}}x)\varphi(x,y)$, for internal $\varphi$ with any (possibly nonstandard) parameters.
- (S) $(\forall x^{\text{st}})(\exists^{\text{st}}y)(\forall^{\text{st}}z)\big((z \in x \wedge \varphi(z)) \leftrightarrow z \in y\big)$, for any formula $\varphi$.
- (T) $(\forall^{\text{st}}x)\varphi(x,t) \to (\forall x)\varphi(x,t)$, where $\varphi$ is internal, $t$ captures *all* parameters of $\varphi$, and $t$ is standard.

The system $\mathsf{IST}$ is (the internal system) $\mathsf{ZFC}$ extended with the aforementioned external axioms; The former is a conservative extension of $\mathsf{ZFC}$ for the internal language, as proved in [16].

In [4], the authors study Gödel's system $\mathsf{T}$ extended with special cases of the external axioms of $\mathsf{IST}$. In particular, they consider nonstandard extensions of the (internal) systems $\mathsf{E\text{-}HA}^\omega$ and $\mathsf{E\text{-}PA}^\omega$, respectively *Heyting and Peano arithmetic in all finite types and the axiom of extensionality*. We refer to [4, §2.1] for the exact details of these (mainstream in mathematical logic) systems. We do mention that in these systems of higher-order arithmetic, each variable $x^\rho$ comes equipped with a superscript denoting its type, which is however often implicit. As to the coding of multiple variables, the type $\rho^*$ is the type of finite sequences of type $\rho$, a notational device used in [4] and this paper; Underlined variables $\underline{x}$ consist of multiple variables of (possibly) different type.

In the next section, we introduce the system $\mathsf{P}$ assuming familiarity with the higher-type framework of Gödel's $\mathsf{T}$ (See e.g. [4, §2.1], [2], [14], or [24]).

### 2.2. The system P.

In this section, we introduce the system $\mathsf{P}$. We first discuss some of the external axioms studied in [4]. First of all, Nelson's axiom *Standard part* is weakened to $\mathsf{HAC_{int}}$ as follows:

$$(\forall^{\text{st}}x^\rho)(\exists^{\text{st}}y^\tau)\varphi(x,y) \to (\exists^{\text{st}}F^{\rho\to\tau^*})(\forall^{\text{st}}x^\rho)(\exists y^\tau \in F(x))\varphi(x,y), \tag{$\mathsf{HAC_{int}}$}$$

where $\varphi$ is any internal formula. Note that $F$ only provides a *finite sequence* of witnesses to $(\exists^{\text{st}}y)$, explaining its name *Herbrandized Axiom of Choice*. Secondly, Nelson's axiom idealisation $\mathsf{I}$ appears in [4] as follows:

$$(\forall^{\text{st}}x^{\sigma^*})(\exists y^\tau)(\forall z^\sigma \in x)\varphi(z,y) \to (\exists y^\tau)(\forall^{\text{st}}x^\sigma)\varphi(x,y), \tag{$\mathsf{I}$}$$

[3] The superscript 'fin' in (I) means that $x$ is finite, i.e. its number of elements are bounded by a natural number.

where $\varphi$ is again an internal formula. Finally, as in [4, Def. 6.1], we have the following definition, where $\mathsf{E\text{-}PA}^{\omega}$ is just Peano arithmetic in al finite types with the axiom of extensionality, and $\mathsf{E\text{-}PA}^{\omega*}$ is a definitional extension of the latter with extra types for finite sequences of objects (See [4, §2.1] and Section 2.3).

**Definition 2.1.** The set $\mathcal{T}^*$ is defined as the collection of all the constants in the language of $\mathsf{E\text{-}PA}^{\omega*}$. The system $\mathsf{E\text{-}PA}^{\omega*}_{\mathrm{st}}$ is defined as $\mathsf{E\text{-}PA}^{\omega*} + \mathcal{T}^*_{\mathrm{st}} + \mathsf{IA}^{\mathrm{st}}$, where $\mathcal{T}^*_{\mathrm{st}}$ consists of the following axiom schemas.

(1) The schema[4] $\mathrm{st}(x) \wedge x = y \to \mathrm{st}(y)$,
(2) The schema providing for each closed term $t \in \mathcal{T}^*$ the axiom $\mathrm{st}(t)$.
(3) The schema $\mathrm{st}(f) \wedge \mathrm{st}(x) \to \mathrm{st}(f(x))$.

The external induction axiom $\mathsf{IA}^{\mathrm{st}}$ is as follows.

$$\big(\Phi(0) \wedge (\forall^{\mathrm{st}} n^0)(\Phi(n) \to \Phi(n+1))\big) \to (\forall^{\mathrm{st}} m^0)\Phi(m). \tag{$\mathsf{IA}^{\mathrm{st}}$}$$

The nonstandard system $\mathsf{P} \equiv \mathsf{E\text{-}PA}^{\omega*}_{\mathrm{st}} + \mathsf{HAC}_{\mathsf{int}} + \mathsf{I} + \mathsf{IA}^{\mathrm{st}}$ is connected to $\mathsf{E\text{-}PA}^{\omega*}$ by Theorem 2.3. The superscript '$S_{\mathrm{st}}$' in the latter is the syntactic translation defined as follows in [4, Def. 7.1].

**Definition 2.2.** If $\Phi(\underline{a})$ and $\Psi(\underline{b})$ in the language of $\mathsf{P}$ have interpretations

$$\Phi(\underline{a})^{S_{\mathrm{st}}} \equiv (\forall^{\mathrm{st}} \underline{x})(\exists^{\mathrm{st}} \underline{y})\varphi(\underline{x}, \underline{y}, \underline{a}) \text{ and } \Psi(\underline{b})^{S_{\mathrm{st}}} \equiv (\forall^{\mathrm{st}} \underline{u})(\exists^{\mathrm{st}} \underline{v})\psi(\underline{u}, \underline{v}, \underline{b}), \tag{2.1}$$

then they interact as follows with the logical connectives by [4, Def. 7.1]:

(i) $\psi^{S_{\mathrm{st}}} := \psi$ for atomic internal $\psi$.
(ii) $\big(\mathrm{st}(z)\big)^{S_{\mathrm{st}}} := (\exists^{\mathrm{st}} x)(z = x)$.
(iii) $(\neg\Phi)^{S_{\mathrm{st}}} := (\forall^{\mathrm{st}} \underline{Y})(\exists^{\mathrm{st}} \underline{x})(\forall \underline{y} \in \underline{Y}[\underline{x}])\neg\varphi(\underline{x}, \underline{y}, \underline{a})$.
(iv) $(\Phi \vee \Psi)^{S_{\mathrm{st}}} := (\forall^{\mathrm{st}} \underline{x}, \underline{u})(\exists^{\mathrm{st}} \underline{y}, \underline{v})[\varphi(\underline{x}, \underline{y}, \underline{a}) \vee \psi(\underline{u}, \underline{v}, \underline{b})]$.
(v) $\big((\forall z)\Phi\big)^{S_{\mathrm{st}}} := (\forall^{\mathrm{st}} \underline{x})(\exists^{\mathrm{st}} \underline{y})(\forall z)(\exists \underline{y}' \in \underline{y})\varphi(\underline{x}, \underline{y}', z)$.

**Theorem 2.3.** *Let $\Phi(\underline{a})$ be a formula in the language of $\mathsf{E\text{-}PA}^{\omega*}_{\mathrm{st}}$ and suppose $\Phi(\underline{a})^{S_{\mathrm{st}}} \equiv \forall^{\mathrm{st}} \underline{x}\, \exists^{\mathrm{st}} \underline{y}\, \varphi(\underline{x}, \underline{y}, \underline{a})$. If $\Delta_{\mathsf{int}}$ is a collection of internal formulas and*

$$\mathsf{P} + \Delta_{\mathsf{int}} \vdash \Phi(\underline{a}), \tag{2.2}$$

*then one can extract from the proof a sequence of closed terms $t$ in $\mathcal{T}^*$ such that*

$$\mathsf{E\text{-}PA}^{\omega*} + \Delta_{\mathsf{int}} \vdash\ \forall \underline{x}\, \exists \underline{y} \in \underline{t}(\underline{x})\ \varphi(\underline{x}, \underline{y}, \underline{a}). \tag{2.3}$$

*Proof.* Immediate by [4, Theorem 7.7]. □

The proofs of the soundness theorems in [4, §5-7] provide an algorithm to obtain the term $t$ from the theorem. The following corollary is only mentioned in [4] for Heyting arithmetic, but is also valid for Peano arithmetic.

**Corollary 2.4.** *If for internal $\psi$ the formula $\Phi(\underline{a}) \equiv (\forall^{\mathrm{st}} \underline{x})(\exists^{\mathrm{st}} \underline{y})\psi(\underline{x}, \underline{y}, \underline{a})$ satisfies* (2.2)*, then $(\forall \underline{x})(\exists \underline{y} \in t(\underline{x}))\psi(\underline{x}, \underline{y}, \underline{a})$ is proved in the corresponding formula* (2.3).

*Proof.* Clearly, if for $\psi$ and $\Phi$ as given we have $\Phi(\underline{a})^{S_{\mathrm{st}}} \equiv \Phi(\underline{a})$, then the corollary follows immediately from the theorem. A tedious but straightforward verification using the clauses (i)-(v) from Definition 2.2 establishes that indeed $\Phi(\underline{a})^{S_{\mathrm{st}}} \equiv \Phi(\underline{a})$. This verification may also be found in [19, §2] or [20, §2.1]. □

[4]The language of $\mathsf{E\text{-}PA}^{\omega*}_{\mathrm{st}}$ contains a symbol $\mathrm{st}_\sigma$ for each finite type $\sigma$, but the subscript is always omitted. Hence $\mathcal{T}^*_{\mathrm{st}}$ is an *axiom schema* and not an axiom.

With regard to notation, for the rest of this paper, a *normal form* refers to a formula of the form $(\forall^{\rm st} x)(\exists^{\rm st} y)\varphi(x, y)$ for $\varphi$ internal.

The previous theorems do not really depend on the presence of full Peano arithmetic; It is an easy verification that the proof of [4, Theorem 7.7] goes through for any fragment of $\textsf{E-PA}^{\omega*}$ which includes $\textsf{EFA}$, sometimes also called $\textsf{I}\Delta_0 + \textsf{EXP}$. In particular, the exponential function is (all what is) required to 'easily' manipulate finite sequences. It should be a straightforward verification to show that the below proofs also go through for $\textsf{E-PA}^{\omega}$ replaced by $\textsf{E-PRA}^{\omega}$ from [13, §2].

Finally, we point out an an interesting property of the previous theorem.

**Remark 2.5.** Consider Theorem 2.3 with $\Delta_{\textsf{int}} \equiv (\exists^2)$ defined as follows:

$$(\exists \varphi^2)(\forall f^1)\big[(\exists n^0)(f(n) = 0) \leftrightarrow \varphi(f) = 0\big]. \tag{$\exists^2$}$$

Note that the resulting term $t$ in (2.3) *is still a term from Gödel's* $\textsf{T}$, i.e. does not involve the Turing jump functional $\varphi$ from $(\exists^2)$, as the latter principle is internal. Hence, even though $\textsf{P}+(\exists^2)$ involves the Turing jump functional, the terms resulting from term extraction as in Theorem 2.3 *are computable* (in the sense of Gödel's $\textsf{T}$).

On the other hand, consider $\textsf{P} + \textsf{STJ}$, where the latter is the *external* principle:

$$(\exists^{\rm st} \varphi^2)\big[(\forall f^1)\big((\exists n^0)(f(n) = 0) \leftrightarrow \varphi(f) = 0\big)\big], \tag{STJ}$$

i.e. the claim that the Turing jump functional *is standard*. If $\textsf{P} + \textsf{STJ}$ proves $(\forall^{\rm st} x)(\exists^{\rm st} y)\psi(x, y)$ for internal $\psi$, then $\textsf{P}$ proves the normal form

$$(\forall^{\rm st} x, \varphi^2)(\exists^{\rm st} y)\big[\textsf{TJ}(\varphi) \to \psi(x, y)\big], \tag{2.4}$$

where $\textsf{TJ}(\varphi)$ is the internal formula in square brackets in $\textsf{STJ}$. Applying Corollary 2.4 to (2.4), we obtain a term $t$ such that $\textsf{E-PA}^{\omega*}$ proves

$$(\forall x, \varphi^2)(\exists y \in t(x, \varphi))\big[\textsf{TJ}(\varphi) \to \psi(x, y)\big],$$

i.e. the term resulting from term extraction *does* involve the Turing jump functional now. In conclusion, the *internal* principles $\Delta_{\textsf{int}}$ in Theorem 2.3 may state the existence of non-computable objects *without affecting the term $t$ in* (2.3). However, *external* principles like $\textsf{TSJ}$ do affect the term extraction process. Or: Only standard objects are relevant to the term extraction process (as also explicitly noted in the proof of [4, Theorem 7.7]).

2.3. **Notations.** We finish this section with some remarks on notation. First of all, we mostly follow Nelson's notations, as sketched now.

**Remark 2.6** (Nonstandard Analysis)**.** We write $(\forall^{\rm st} x^\tau)\Phi(x^\tau)$ and $(\exists^{\rm st} x^\sigma)\Psi(x^\sigma)$ as short for $(\forall x^\tau)\big[\mathrm{st}(x^\tau) \to \Phi(x^\tau)\big]$ and $(\exists x^\sigma)\big[\mathrm{st}(x^\sigma) \wedge \Psi(x^\sigma)\big]$. We also write $(\forall x^0 \in \Omega)\Phi(x^0)$ and $(\exists x^0 \in \Omega)\Psi(x^0)$ as short for $(\forall x^0)\big[\neg\mathrm{st}(x^0) \to \Phi(x^0)\big]$ and $(\exists x^0)\big[\neg\mathrm{st}(x^0) \wedge \Psi(x^0)\big]$. Furthermore, if $\neg\mathrm{st}(x^0)$ (resp. $\mathrm{st}(x^0)$), we also say that $x^0$ is 'infinite' (resp. finite) and write '$x^0 \in \Omega$'. A formula $A$ is 'internal' if it does not involve st, and $A^{\rm st}$ is defined from $A$ by appending 'st' to all quantifiers (except bounded number quantifiers).

Secondly, the usual extensional notion of equality is used in $\textsf{P}$.

**Remark 2.7** (Equality)**.** The system $\textsf{P}$ includes equality between natural numbers '$=_0$' as a primitive. Equality '$=_\tau$' for type $\tau$-objects $x, y$ is then defined as:

$$[x =_\tau y] \equiv (\forall z_1^{\tau_1} \dots z_k^{\tau_k})[x z_1 \dots z_k =_0 y z_1 \dots z_k] \tag{2.5}$$

if the type $\tau$ is composed as $\tau \equiv (\tau_1 \to \ldots \to \tau_k \to 0)$. In the spirit of Nonstandard Analysis, we define 'approximate equality $\approx_\tau$' as follows:

$$[x \approx_\tau y] \equiv (\forall^{\mathrm{st}} z_1^{\tau_1} \ldots z_k^{\tau_k})[x z_1 \ldots z_k =_0 y z_1 \ldots z_k] \tag{2.6}$$

with the type $\tau$ as above. The system $\mathsf{P}$ includes the *axiom of extensionality* for all $\varphi^{\rho\to\tau}$ as follows:

$$(\forall x^\rho, y^\rho)\big[x =_\rho y \to \varphi(x) =_\tau \varphi(y)\big]. \tag{$\mathsf{E}$}$$

However, as noted in [4, p. 1973], the so-called axiom of *standard* extensionality $(\mathsf{E})^{\mathrm{st}}$ is problematic and cannot be included in $\mathsf{P}$. Finally, we need the following explicit version of the axiom of extensionality $(\mathsf{E})$.

$$(\forall f^1, g^1)(\overline{f}\Xi(f,g) =_0 \overline{g}\Xi(f,g) \to Y(f) =_0 Y(g)). \tag{$\mathsf{EXT}(\Xi, Y)$}$$

We say that $\Xi^2$ is an *extensionality functional* of the functional $Y^2$.

Thirdly, we use the usual notations related to sequences from [4, §2.1].

**Remark 2.8** (Finite sequences)**.** The system $\mathsf{E}\text{-}\mathsf{PA}^{\omega*}$ has a dedicated type for 'finite sequences of objects of type $\rho$', namely $\rho^*$. Since the usual coding of pairs of numbers goes through in $\mathsf{E}\text{-}\mathsf{PA}^{\omega*}$, we shall not always distinguish between 0 and $0^*$; See e.g. the definition of the Gandy-Hyland functional in Section 1.

For sequences $s^{\rho^*}, t^{\rho^*}$, we denote by '$s * t$' the concatenation of $s$ and $t$. For a sequence $s^{\rho^*}$, we denote by $|s|$ its length, and by $\overline{s}N = (s(0)\ldots s(N))$ for $N^0 < |s|$. For a sequence $\alpha^{0\to\rho}$, we also write $\overline{\alpha}N = (\alpha(0)\ldots\alpha(N))$ for any $N^0$. By way of shorthand, $r^\rho \in R^{\rho^*}$ abbreviates $(\exists i < |R|)(R(i) =_\rho r)$.

Finally, we introduce some (strictly speaking) abuse of notation.

**Remark 2.9** (Set-theoretic notation)**.** As is clear from the previous remark, we sometimes use intuitive set-theoretic notation in this paper, although $\mathsf{P}$ only involves functionals. First of all, we assume that sets $X^1$ are given by their characteristic functions $f_X^1$, i.e. $(\forall x^0)[x \in X \leftrightarrow f_X(x) = 1]$. Secondly, the notation '$Y^2 \in C$' means that $Y$ is continuous on Baire space 'as usual' given by (4.1). A formula $(\forall^{\mathrm{st}} Y^2 \in C)(\ldots)$ is thus shorthand for $(\forall Y^2)\big([\mathrm{st}(Y) \wedge Y \in C] \to \ldots\big)$; Note in particular that no mention *whatsoever* of $(4.1)^{\mathrm{st}}$ is made, or will be made below. Thirdly, we sometimes block quantifiers together to save space; In this way, the formula $\big(\forall (Y^2 \in C, Z^2) \in \Psi\big)(\ldots)$ for some $\Psi^{2^*}$, is shorthand for

$$(\forall Z^2)(\forall Y^2)\big(\big[Y \in C \wedge (\exists j < |\Psi|)(\Psi(j) =_2 Y) \wedge (\exists i < |\Psi|)(\Psi(i) =_2 Z)\big] \to \ldots\big),$$

which arguably saves space.

## 3. Preliminaries

In this section, we prove some preliminary results needed below. In particular, we study two weak fragments of the *Standard Part principle* of $\mathsf{IST}$ in Section 3.1. Furthermore, in Section 3.2, we derive a version of bar induction for *external* formulas from a weak fragment of the *Standard Part principle* of $\mathsf{IST}$.

### 3.1. Fragments of the Standard Part principle.

In this section, we discuss two fragments of the *Standard Part principle* of $\mathsf{IST}$ essential to our results. Recall that *weak König's lemma*, denoted $\mathsf{WKL}$, is the statement that every infinite binary tree has a path (See e.g. [21, IV]).

3.1.1. *Nonstandard weak König's lemma.* In this section, we establish that the following fragment of the *Standard part principle* is a nonstandard version of WKL.

$$(\forall f^1 \leq_1 1)(\exists^{\rm st} g^1 \leq_1 1)(f \approx_1 g). \tag{STP}$$

The function $g^1$ from STP is called the *standard part* of $f^1$.

**Theorem 3.1.** *The system* P *proves that* STP *is equivalent to*

$$(\forall T^1 \leq_1 1)\big[(\forall^{\rm st} n)(\exists \beta^0)(|\beta| = n \wedge \beta \in T) \\ \rightarrow (\exists^{\rm st} \alpha^1 \leq_1 1)(\forall^{\rm st} n^0)(\overline{\alpha} n \in T)\big]. \tag{3.1}$$

*where '* $T \leq_1 1$*' denotes that* $T$ *is a binary tree. Over* P, STP *is also equivalent to*

$$(\forall f^1)(\exists^{\rm st} g^1)\big((\forall^{\rm st} n^0)(\exists^{\rm st} m^0)(f(n) = m) \rightarrow f \approx_1 g\big). \tag{3.2}$$

*Proof.* Assume STP and apply overflow to $(\forall^{\rm st} n)(\exists \beta^0)(|\beta| = n \wedge \beta \in T)$ (See [4, Prop. 3.3]) to obtain $\beta_0 \in T$ with nonstandard length $|\beta_0|$. Now apply STP to $\beta_0 * 00\ldots$ to obtain a *standard* path through $T$. For the reverse direction, let $f^1$ be a binary sequence, and define a tree $T_f$ which contains all initial segments of $f$. Now apply (3.1) to obtain STP.

For the remaining equivalence, the implication (3.2) $\rightarrow$ STP is trivial, and for the reverse implication, fix $f^1$ such that $(\forall^{\rm st} n)(\exists^{\rm st} m) f(n) = m$ and let $h^1$ be such that $(\forall n, m)(f(n) = m \leftrightarrow h(n, m) = 1)$. Applying $\mathsf{HAC_{int}}$ to the former formula, there is standard $\Phi^{0 \rightarrow 0^*}$ such that $(\forall^{\rm st} n)(\exists m \in \Phi(n)) f(n) = m$, and define $\Psi(n) := \max_{i < |\Phi(n)|} \Phi(n)(i)$. Now define the sequence $\alpha_0$ as follows: $\alpha_0(0) := h(0, 0)$, $\alpha_0(1) := h(0, 1)$, $\ldots$, $\alpha_0(\Psi(0)) := h(0, \Psi(0))$, $\alpha_0(\Psi(0) + 1) := h(1, 0)$, $\alpha_0(\Psi(0) + 2) := h(1, 1)$, $\ldots$, $\alpha_0(\Psi(0) + \Psi(1)) := h(1, \Psi(1))$, et cetera. Now let $\beta_0^1 \leq_1 1$ be the standard part of $\alpha_0$ provided by STP and define $g(n) := (\mu m \leq \Psi(n))\big[\beta_0(\sum_{i=0}^{n-1} \Psi(i) + m) = 1\big]$. By definition, $g^1$ is standard and $f \approx_1 g$. □

The function $g^1$ from (3.2) is also called the *standard part* of $f^1$. We now show that STP follows from the nonstandard uniform continuity of all type two functionals on Cantor space. Note that the principle NUC in the theorem contradicts classical mathematics, as the latter involves *discontinuous* functionals.

**Theorem 3.2.** *The axiom* STP *can be proved in* P *plus the axiom*

$$(\forall^{\rm st} Y^2)(\forall f^1, g^1 \leq_1 1)(f \approx_1 g \rightarrow Y(f) =_0 Y(g)). \tag{NUC}$$

*Proof.* First of all, note that NUC implies by definition (and classical logic) that

$$(\forall^{\rm st} Y^2)(\forall f^1, g^1 \leq_1 1)(\exists^{\rm st} N^0)(\overline{g} N =_0 \overline{f} N \rightarrow Y(f) =_0 Y(g)). \tag{3.3}$$

Applying idealisation I to (3.3), we obtain that

$$(\forall^{\rm st} Y^2)(\exists^{\rm st} N_0^0)(\forall f^1, g^1 \leq_1 1)(\overline{g} N_0 =_0 \overline{f} N_0 \rightarrow Y(f) =_0 Y(g)). \tag{3.4}$$

Thus, every standard $Y^2$ is bounded on Cantor space; In particular, for $Y$ and $N_0$ as in (3.4) we have $(\forall f^1 \leq_1 1)(Y(f) \leq \max_{\sigma \leq_{0^*} 1 \wedge |\sigma| = N_0}) Y(\sigma * 00\ldots)$, which yields a *standard* bound. Now consider the contraposition of (3.1) for some fixed $T \leq_1 1$, and assume $(\forall^{\rm st} \alpha^1 \leq_1 1)(\exists^{\rm st} n^0)(\overline{\alpha} n \notin T)$. Similar to the second part of the previous proof, applying $\mathsf{HAC_{int}}$ yields a standard functional $Y_0^2$ such that $(\forall^{\rm st} \alpha^1 \leq_1 1)(\exists i \leq Y_0(\alpha))(\overline{\alpha} i \notin T)$. By the previous, $Y_0$ is bounded on Cantor space, yielding $(\exists^{\rm st} k^0)(\forall \beta^1 \leq_1 1)(\exists i \leq k)(\overline{\beta} i \notin T)$, and STP follows from Theorem 3.1. □

As explained in the introduction, the theme of this paper is the extraction of relative computability results from theorems of Nonstandard Analysis. We now provide the first example of this theme in Corollary 3.3, based on the proof of $\mathsf{NUC} \to \mathsf{STP}$ in the previous theorem. The following definitions are relevant.

$$(\forall Y^2)(\forall f^1, g^1 \leq_1 1)(\overline{f}\Omega(Y) = \overline{g}\Omega(Y) \to Y(f) = Y(g)). \tag{$\mathsf{MUC}(\Omega)$}$$

$$(\forall g^2, T^1 \leq_1 1)\big[(\forall \alpha^1 \in \Theta(g)(2))(\alpha \leq_1 1 \to \overline{\alpha}g(\alpha) \not\in T) \to (\forall \beta \leq_1 1)(\exists i \leq_0 \Theta(g)(1))(\overline{\beta}i \not\in T)\big]. \tag{$\mathsf{SCF}(\Theta)$}$$

The functional $\Omega^3$ as in $\mathsf{MUC}(\Omega)$ is called the *fan functional* and is essentially a conservative extension of $\mathsf{WKL}$ for the second-order language (See [13, Prop. 3.15]). We now show that $\Theta$ as in $\mathsf{SCF}(\Theta)$ is a 'special case' of the fan functional; Thus, we sometimes refer to the functional $\Theta$ as the *special* fan functional.

**Corollary 3.3.** *From the proof in* $\mathsf{P}$ *that* $\mathsf{NUC} \to \mathsf{STP}$*, a term* $t$ *can be extracted such that* $\mathsf{E\text{-}PA}^{\omega*}$ *proves* $(\forall \Omega^3)\big[\mathsf{MUC}(\Omega) \to \mathsf{SCF}(t(\Omega))\big]$.

*Proof.* By the proof of the theorem, $\mathsf{NUC}$ is equivalent to the normal form (3.4), which we abbreviate as $(\forall^{\mathrm{st}} Y^2)(\exists^{\mathrm{st}} N^0)A(Y, N)$. Now note that (3.1) implies

$$(\forall T^1 \leq_1 1)(\forall^{\mathrm{st}} g^2)\big[(\forall^{\mathrm{st}} \alpha \leq_1 1)(\overline{\alpha}g(\alpha) \not\in T) \to (\exists^{\mathrm{st}} k^0)(\forall \beta \leq_1 1)(\exists i \leq k)(\overline{\beta}i \not\in T)\big]. \tag{3.5}$$

Bringing standard quantifiers outside, we obtain that

$$(\forall^{\mathrm{st}} g^2)(\forall T^1 \leq_1 1)(\exists^{\mathrm{st}} k^0, \alpha^1 \leq_1 1)\big[(\overline{\alpha}g(\alpha) \not\in T) \to (\forall \beta \leq_1 1)(\exists i \leq k)(\overline{\beta}i \not\in T)\big].$$

Applying idealisation $\mathsf{I}$, we pull the standard quantifiers to the front as follows:

$$(\forall^{\mathrm{st}} g^2)(\exists^{\mathrm{st}} w^{1^*})(\forall T^1 \leq_1 1)(\exists \alpha^1 \leq_1 1, k^0 \in w)\big[(\overline{\alpha}g(\alpha) \not\in T) \to (\forall \beta \leq_1 1)(\exists i \leq k)(\overline{\beta}i \not\in T)\big],$$

which we abbreviate as $(\forall^{\mathrm{st}} g^2)(\exists^{\mathrm{st}} w^{1^*})B(g, w)$. Hence, the proof of $\mathsf{NUC} \to \mathsf{STP}$ yields a proof of $(\forall^{\mathrm{st}} Y^2)(\exists^{\mathrm{st}} N^0)A(Y, N) \to (\forall^{\mathrm{st}} g^2)(\exists^{\mathrm{st}} w^{1^*})B(g, w)$, which yields

$$(\forall^{\mathrm{st}} \Omega^3)\big[(\forall Y^2)A(Y, \Omega(Y)) \to (\forall^{\mathrm{st}} g^2)(\exists^{\mathrm{st}} w^{1^*})B(g, w)\big],$$

by strengthening the antecedent. Bringing all standard quantifiers up front:

$$(\forall^{\mathrm{st}} \Omega^3, g^2)(\exists^{\mathrm{st}} w^{1^*})\big[(\forall Y^2)A(Y, \Omega(Y)) \to B(g, w)\big]; \tag{3.6}$$

Applying Corollary 2.4 to $\mathsf{P} \vdash$ (3.6), we obtain a term $t$ such that $\mathsf{E\text{-}PA}^{\omega*}$ proves

$$(\forall \Omega^3, g^2)(\exists w \in t(\Omega, g))\big[(\forall Y^2)A(Y, \Omega(Y)) \to B(g, w)\big], \tag{3.7}$$

Clearly, the antecedent of (3.7) expresses that $\Omega^3$ is the fan functional, while $t(\Omega, g)$ is essentially the functional $\Theta$ from $\mathsf{SCF}(\Theta)$, and we are done. $\square$

For comparison, we provide a 'direct' proof of the previous corollary.

**Corollary 3.4.** *There is a term* $t$ *such that* $\mathsf{P}$ *proves* $(\forall \Omega^3)\big[\mathsf{MUC}(\Omega) \to \mathsf{SCF}(t(\Omega))\big]$.

*Proof.* Note that $\Theta(g)$ as in $\mathsf{SCF}(\Theta)$ has to provide a natural number and a finite sequence of binary sequences. The natural number, say $\Theta(g)(1)$, is defined as $\max_{|\sigma| = \Omega(g) \wedge \sigma \leq_{0^*} 1} g(\sigma * 00\dots)$ and the finite sequence of binary sequences, say $\Theta(g)(2)$, consists of all $\tau * 00\dots$ where $|\tau| = \Theta(g)(1) \wedge \tau \leq_{0^*} 1$. We now have

$$(\forall \beta \leq_1 1)(\beta \in \Theta(g)(2) \to \overline{\beta}g(\beta) \not\in T) \to (\forall \gamma \leq_1 1)(\exists i \leq \Theta(g)(1))(\overline{\gamma}i \not\in T). \tag{3.8}$$

Indeed, suppose the antecendent of (3.8) holds. Now take $\gamma_0 \leq_1 1$, and note that $\beta_0 = \overline{\gamma_0}\Theta(g)(1) * 00\cdots \in \Theta(g)(2)$, implying $\overline{\beta_0}g(\beta_0) \notin T$. But $g(\alpha) \leq \Theta(g)(1)$ for all $\alpha \leq_1 1$, by the definition of $\Omega$, implying that $\overline{\gamma_0}g(\beta_0) = \overline{\beta_0}g(\beta_0) \notin T$ by the definition of $\beta_0$, and the consequent of (3.8) follows. □

3.1.2. *A 'computable' fragment of the Standard Part principle.* In this section, we discuss the Standard Part principle $\Omega$-CA, a very practical consequence of $\mathsf{HAC}_{\mathsf{int}}$. Intuitively speaking, $\Omega$-CA expresses that we can obtain the standard part (in casu $G$) of $\Omega$-*invariant* nonstandard objects (in casu $F(\cdot, M)$).

**Definition 3.5.** [$\Omega$-invariance] Let $F^{(\sigma\times 0)\to 0}$ be standard and fix $M^0 \in \Omega$. Then $F(\cdot, M)$ is **$\Omega$-invariant** if

$$(\forall^{\text{st}} x^\sigma)(\forall N^0 \in \Omega)\big[F(x, M) =_0 F(x, N)\big]. \tag{3.9}$$

**Principle 3.6** ($\Omega$-CA)**.** Let $F^{(\sigma\times 0)\to 0}$ be standard and fix $M \in \Omega$. For $\Omega$-invariant $F(\cdot, M)$, there is standard $G^{\sigma\to 0}$ such that

$$(\forall^{\text{st}} x^\sigma)(\forall N^0 \in \Omega)\big[G(x) =_0 F(x, N)\big]. \tag{3.10}$$

The axiom $\Omega$-CA provides the standard part of a nonstandard object, if the latter is *independent of the choice of infinite number* used in its definition.

**Theorem 3.7.** *The system* P *proves* $\Omega$-CA.

*Proof.* Assuming $F(\cdot, M^0)$ is $\Omega$-invariant, i.e. we have

$$(\forall^{\text{st}} x^\sigma)(\forall N^0, M^0 \in \Omega)\big[F(x, M) =_0 F(x, N)\big], \tag{3.11}$$

it is easy to obtain (e.g. via minimisation present in P) that

$$(\forall^{\text{st}} x^\sigma)(\exists^{\text{st}} k^0)(\forall N^0, M^0 \geq k)\big[F(x, M) =_0 F(x, N)\big]. \tag{3.12}$$

Indeed, note that (3.11) trivially implies (take any $m \in \Omega$) that

$$(\forall^{\text{st}} x^\sigma)(\exists m)(\forall N^0, M^0 \geq m)\big[F(x, M) =_0 F(x, N)\big],$$

and obtain the least such $m$ using the (internal) minimisation axioms present in P. By (3.11), this least number must be standard, and we obtain (3.12). If one wishes to avoid induction, one can apply the so-called *underspill*[5] principle (See [4, Prop. 5.11]) to (3.11) to obtain (3.12) directly.

Now apply $\mathsf{HAC}_{\mathsf{int}}$ to (3.12) to obtain standard $\Phi^{\sigma\to 0^*}$ such that

$$(\forall^{\text{st}} x^\sigma)(\exists k^0 \in \Phi(x))(\forall N^0, M^0 \geq k)\big[F(x, M) =_0 F(x, N)\big].$$

Define $\Psi(x) := \max_{i<|\Phi(x)|} \Phi(x)(i)$ and note that

$$(\forall^{\text{st}} x^\sigma)(\forall N^0, M^0 \geq \Psi(x))\big[F(x, M) =_0 F(x, N)\big].$$

Finally, define $G(x) := F(x, \Psi(x))$ and note that the latter is as in $\Omega$-CA. □

**Remark 3.8.** It is straightforward to verify that Theorem 3.7 also holds if the quantifier '$(\forall^{\text{st}} x^\sigma)$' in (3.9) and (3.10) is restricted to '$(\forall^{\text{st}} x^\sigma)(C(x) \to \dots)$', where $C$ is any internal formula. We shall also refer to this slight extension as $\Omega$-CA. The axiom $\Omega$-CA can also be generalised to $F^{(\sigma\times 0)\to\tau}$ using the approximate equality '$\approx_\tau$' defined in Remark 2.7. However, the above version suffices for our purposes.

[5]We shall henceforth freely invoke the aforementioned underspill principle when working in P.

3.2. **External bar induction.** In this section, we derive a version of bar induction from $\mathsf{STP}$. The former is a generalisation of the well-known principle of (mathematical) induction of arithmetic. While induction takes place 'along the natural numbers' (or any well-order), bar induction takes place 'down a tree'. As an example, we consider the following principle.

**Principle 3.9** ($\mathsf{BI}_0$). *For internal quantifier-free $Q(x^0)$, if*

$$(\forall\alpha^1)(\exists n)Q(\overline{\alpha}n) \wedge (\forall t^0)\big[(\forall x^0)Q(t * \langle x\rangle) \to Q(t)\big] \tag{3.13}$$

*then we have $Q(\langle\rangle)$.*

Intuitively speaking, bar induction $\mathsf{BI}_0$ expresses that we may conclude $Q(x)$ for $x = \langle\rangle$ from the fact that $Q$ is implied 'downwards' from child nodes to parent nodes (second conjunct of (3.13)) and that $Q$ holds eventually along any path (first conjunct of (3.13)). On a technical note, $\mathsf{BI}_0$ is essentially $\mathsf{BI}_{\rm qf}$ from [23, p. 78] for $P(n) \equiv Q(n)$ quantifier-free. We now prove $\mathsf{BI}_0^{\rm st}$ form $\mathsf{STP}$.

**Theorem 3.10.** *In $\mathsf{P} + \mathsf{STP}$, we have $\mathsf{BI}_0^{\rm st}$.*

*Proof.* Assume $(3.13)^{\rm st}$ and suppose we have $\neg Q(\langle\rangle)$. Now define $F(x, M) := (\mu m \leq M)\neg Q(x * \langle m\rangle)$ and put $G(0) := F(\langle\rangle, M)$ and $G(n + 1) := F(G(0) * \cdots * G(n), M)$. By $(3.13)^{\rm st}$ and the assumption $\neg Q(\langle\rangle)$, $G(0)$ is standard. Furthermore, we also have that $G(n+1)$ is standard if $G(k)$ is standard, for standard $n$ and $k \leq n$, by $(3.13)^{\rm st}$, implying that $G(n)$ is standard for all standard $n$ by external induction $\mathsf{IA}^{\rm st}$. This in turn implies that $\neg Q(G(0) * \cdots * G(k))$ for standard $k$, by quantifier-free induction and $(3.13)^{\rm st}$. Now consider the sequence $\beta^1 = G(0)*G(1)*G(2)*\ldots$ and take its standard part $\alpha^1$. Finally, apply the first conjunct of $(3.13)^{\rm st}$ and obtain a contradiction. $\square$

The previous theorem is not that surprising: $\mathsf{STP}$ is the nonstandard version of weak König's lemma by Theorem 3.1, the latter lemma is equivalent to a version of dependent choice (See [21, VIII.2.5]), and bar induction is a version of the latter.

Nonstandard Analysis also has a distinct kind of induction, called *external induction*. We consider the following example.

**Principle 3.11** ($\mathsf{ExInd}$). *For standard $F^{(0\times 0)\to 0}$ and $M \in \Omega$, if*

$$\text{st}(F(0, M)) \wedge (\forall^{\rm st} n)[\text{st}(F(n, M)) \to \text{st}(F(n + 1, M))], \tag{3.14}$$

*then $(\forall^{\rm st} n)(\text{st}(F(n, M))$.*

Intuitively speaking, ($\mathsf{ExInd}$) tells us that we may use induction on the new standardness predicate *along the standard numbers* (and obviously not along the numbers). Although seemingly more general than normal induction, we now derive $\mathsf{ExInd}$ from the standardness of the recursor constants in $\mathsf{P}$.

**Theorem 3.12.** *The system $\mathsf{P} \setminus \{\mathsf{IA}^{\rm st}\}$ proves* ($\mathsf{ExInd}$).

*Proof.* Assume (3.14) and replace 'st' as follows:

$$(\forall^{\rm st} n)[(\exists^{\rm st} k^0)(F(n, M) \leq k) \to (\exists^{\rm st} l^0)(F(n + 1, M) \leq l),$$

yielding

$$(\forall^{\rm st} n, k)(\exists^{\rm st} l)[F(n, M) \leq k \to F(n + 1, M) \leq l],$$

and $\mathsf{HAC}_{\mathsf{int}}$ yields standard $g$ such that $l = g(n, k)$ in the previous formula. Use primitive recursion to define the *standard* function $h^1$ such that $h(0) := F(0, M)$

and $h(k+1) := g(k, h(k))$. By the definition of $h$, we have $F(n, M) \leq h(n)$ for standard $n$, proved by quantifier-free induction (of the non-external variety). As $h(n)$ is standard for standard $n$, (3.14) implies the consequent of (ExInd). $\square$

Note that the same proof goes through for variations of (ExInd), e.g. if the induction hypothesis involves $(\forall k \leq n)(\mathrm{st}(F(k, M)))$ instead of $\mathrm{st}(F(n, M))$. Note that (ExInd) also follows directly from $\mathsf{IA}^{\mathrm{st}}$, but the latter cannot be included in fragments of P based on E-PRA$^\omega$ (See [13, §2]).

**Corollary 3.13.** *The system* $\mathsf{P} \setminus \{\mathsf{IA}^{\mathrm{st}}\} + \mathsf{STP}$ *proves* $\mathsf{BI}_0^{\mathrm{st}}$.

We now formulate *external bar induction*, which is nothing more than bar induction on the external standardness predicate.

**Principle 3.14** (EBI). *For standard* $F^{(0\times 0)\to 0}$ *and* $M \in \Omega$, *if*

$$(\forall^{\mathrm{st}}\alpha^1)(\exists^{\mathrm{st}} n^0)\big[\mathrm{st}(F(\overline{\alpha}n, M))\big] \tag{3.15}$$

$$\wedge(\forall^{\mathrm{st}} t^0)\big[(\forall^{\mathrm{st}} x^0)(\mathrm{st}(F(t * \langle x\rangle, M))) \to \mathrm{st}(F(t, M))\big] \tag{3.16}$$

*then* $\mathrm{st}(F(\langle\rangle, M))$.

As it happens, external bar induction EBI is not much stronger than normal bar induction, as we now prove EBI from STP.

**Theorem 3.15.** *The system* $\mathsf{P} + \mathsf{STP}$ *proves* EBI.

*Proof.* In Theorem 3.1, we proved that STP is equivalent to (3.1), which is a nonstandard version of WKL. In particular, (3.1) is just the generalisation of $\mathsf{WKL}^{\mathrm{st}}$ to all (possibly nonstandard) binary trees. By [21, Note 1, p. 54, and VIII.2.5], WKL is equivalent to $\Pi_1^0$-$\mathsf{DC}_0$, the axiom of dependent choice restricted to $\Pi_1^0$-formulas; The same holds for the 'usual' axiom of choice $\Pi_1^0$-$\mathsf{AC}_0$. From this equivalence, we immediately obtain that $\mathsf{WKL}^{\mathrm{st}} \leftrightarrow (\Pi_1^0\text{-}\mathsf{DC}_0)^{\mathrm{st}} \leftrightarrow (\Pi_1^0\text{-}\mathsf{AC}_0)^{\mathrm{st}}$ in P. Similar to Theorem 3.1, it is now a straightforward but tedious[6] verification that STP implies (and is equivalent to) the schema expressing that

$$(\forall^{\mathrm{st}} n^0, X^1)(\exists^{\mathrm{st}} Y^1)\eta^{\mathrm{st}}(X, Y, n) \to (\exists^{\mathrm{st}} Z^1)(\forall^{\mathrm{st}} n^0)\eta^{\mathrm{st}}((Z)^n, (Z)_n, n), \tag{3.17}$$

for any $\Pi_1^0$-formula $\eta$ in the language of P. To obtain EBI from (3.17), assume (3.15) and (3.16), and suppose $F(\langle\rangle, M)$ is nonstandard. Now bring (3.16) in the following form

$$(\forall^{\mathrm{st}} t^0)(\exists^{\mathrm{st}} x^0, m^0)(\forall^{\mathrm{st}} l^0)\big[F(t * \langle x\rangle, M) \leq l \to F(t, M) \leq m\big], \tag{3.18}$$

and apply (3.17) to obtain standard $g^1$ such that

$$(\forall^{\mathrm{st}} t^0)(\forall^{\mathrm{st}} l^0)\big[F(t * \langle g(t)(1)\rangle, M) \leq l \to F(t, M) \leq g(t)(2)\big]. \tag{3.19}$$

Now consider the standard sequence $\alpha^1$ defined by $\alpha(0) := g(\langle\rangle)(1)$ and $\alpha(n+1) := g(\overline{\alpha}n)(1)$. By (3.15), there is standard $n$ such that $F(\overline{\alpha}n, M)$ is standard. However, then there is standard $l$ such that $F(\overline{\alpha}n, M) = F(\overline{\alpha}(n-1) * \langle g(\overline{\alpha}(n-1))(1)\rangle, M)$ *by definition* satisfies the antecedent of (3.19). Hence, $F(\overline{\alpha}(n-1), M)$ is also standard by (3.19), and external induction (ExInd) implies that $F(\langle\rangle, M)$ is standard, a contradiction, and EBI follows. $\square$

[6] A 'direct' proof is as follows: Let $A(t, x, m, n, l)$ be the formula in square brackets in (3.18) and let $f^1$ be the characteristic function of $A$. Now let $g$ be the standard part of $f$ and consider the version of (3.18) $\to$ (3.19) with $f$ replaced by $g$. This modification follows from $(\Pi_1^0\text{-}\mathsf{DC}_0)^{\mathrm{st}}$ (as everything is standard), and hence (3.18) $\to$ (3.19) follows from STP and Theorem 3.1.

## 4. The Gandy-Hyland functional in Nonstandard Analysis

In this section, we list our main results concerning the Gandy-Hyland functional and its so-called canonical approximation $H$, both introduced in the first section. As to its provenance, we recall that the $\Gamma$-functional was introduced in [12] as an example of a functional not Kleene-S1-S9-computable over the total continuous functionals, even with the fan functional as an oracle (See [17, §4] or [15, §8]).

Using the results from the previous section, we prove in Section 4.1 that the Gandy-Hyland functional $\Gamma(\cdot)$ equals $H(\cdot, M)$ from Section 1 for all standard inputs and nonstandard $M$; This proof takes place in an extension of the system $\mathsf{P}$ from Section 2.2. From this nonstandard proof, we extract a term from Gödel's $\mathsf{T}$ expressing $\Gamma$ in terms of the special fan functional (See Corollary 3.3) and a modulus-of-continuity functional. Note that this relative computability result *does not involve Nonstandard Analysis*. In Section 4.3 to 4.6, we obtain similar nonstandard and relative computability results. In particular, we study the *weak continuity functional*, which was first introduced in [7].

Finally, we show in Section 4.2 that one can re-obtain the original nonstandard theorem (that the Gandy-Hyland functional $\Gamma(\cdot)$ equals $H(\cdot, M)$ for all standard inputs and nonstandard $M$) from the proof of a certain natural relative computability result, called the *Herbrandisation* of the original nonstandard theorem. In this way, the latter is seen to have the same computational content as its Herbrandisation. The latter can be easily obtained for *any* nonstandard theorem in this paper.

### 4.1. **From Nonstandard Analysis to relative computability.**

In this section, we prove that the functional $H(\cdot, M)$ and $\Gamma(\cdot)$ are equal for standard inputs and nonstandard $M^0$ in an extension of $\mathsf{P}$. From this proof, we extract a term from Gödel's system $\mathsf{T}$ which computes the $\Gamma$-functional as a function of the special fan functional (See Corollary 3.3) and a modulus-of-continuity functional.

To this end, we consider the following functional $G$, a 'less finitary' version of $H$, but which is still primitive recursive by [9, Theorem 18]. We also refer to $G$ as a *canonical approximation* of $\Gamma$.

$$G(Y, s, N) = \begin{cases} Y(s * 00\dots) & |s| \geq N \\ Y(s * 0 * (\lambda n)G(Y, s * (n+1), N)) & \text{otherwise} \end{cases}$$

Now, as noted in the first section, the $\Gamma$-functional corresponds to modified bar recursion of type 0 (See [6, §4]). Since bar recursion holds in the model of all total continuous functionals (See [7, 11]), the easiest way of obtaining $\Gamma$ from $H$ or $G$ seems to be adding the following continuity axiom to $\mathsf{P}$ (Recall Remark 2.7):

$$(\forall^{\mathrm{st}} Y^2 \in C, f^1)(\forall g^1)(f \approx_1 g \to Y(f) =_0 Y(g)), \tag{$\mathsf{NPC}$}$$

where '$Y^2 \in C$' is the (internal) definition of continuity on Baire space as follows:

$$(\forall f^1)(\exists N^0)(\forall g^1)(\overline{g}N =_0 \overline{f}N \to Y(f) =_0 Y(g)). \tag{4.1}$$

As discussed in Remark 4.4, $\mathsf{NPC}$ without the restriction '$Y^2 \in C$' is inconsistent, while $\mathsf{NPC}$ easily[7] follows from the $\mathsf{IST}$ axiom *Transfer*.

[7] Fix a standard $Y^2 \in C$ and standard $f^1$ as in (4.1), and apply (the contraposition of) *Transfer*.

Furthermore, according to [6, p. 167], the role of the continuity principle and bar induction in [6, Theorem 2.5] is *to verify the correctness of the* [*bar recursive*] *witnessing functional*. As was proved in Section 3.2, the principle $\mathsf{STP}$ from Section 3.1 implies a version of bar induction. Hence, we arrive at the following theorem.

**Theorem 4.1.** *In* $\mathsf{P} + \mathsf{NPC} + \mathsf{STP}$*, we have*

$$(\forall^{\mathrm{st}} Y^2 \in C, s^0)(\forall M, N \in \Omega)(H(Y, s, N) =_0 H(Y, s, M)), \tag{4.2}$$

$$(\forall^{\mathrm{st}} Y^2 \in C, s^0)(\forall M, N \in \Omega)(G(Y, s, N) =_0 G(Y, s, M)), \tag{4.3}$$

*i.e. the canonical approximations of* $\Gamma$ *are* $\Omega$*-invariant.*

*Proof.* We sketch the proof of the theorem and then provide a detailed version.

First of all, one obtains external bar induction from $\mathsf{STP}$ as in Theorem 3.10. Secondly, one uses this bar induction to prove that $G(Y, s, M)$ is standard for standard $Y^2 \in C, s^0$ and infinite $M^0$. Thirdly, one applies bar induction again to prove that (4.2) holds for fixed inputs. Fourth, the result for $H(\cdot, M)$ is now straightforward. Intuitively speaking, $H$ and $G$ only really differ in the second case of their definition for elements in the sequence *with infinite index*, which does not matter due to nonstandard continuity. We now provide a detailed proof.

By Theorem 3.10, we may freely use external bar induction $\mathsf{EBI}$. We first prove that $G(\cdot, M)$ is standard for standard input and infinite $M$. To this end, fix standard $Y^2 \in C, s^0$ and $M \in \Omega$, and define $F(x^0, M) := G(Y, s * x, M)$.

To prove (3.15), fix standard $\gamma^1$ and $N \in \Omega$. We have

$$\begin{aligned} F(\overline{\gamma}N, M) &= G(Y, s * \overline{\gamma}N, M) \\ &= Y(s * \overline{\gamma}N * 0 * (\lambda n)G(Y, s * \overline{\gamma}N * (n+1), M)) && (4.4) \\ &= Y(s * \gamma), && (4.5) \end{aligned}$$

where the final step follows by nonstandard continuity $\mathsf{NPC}$ as $s * \gamma \approx_1 \zeta$, where the latter is the sequence in (4.4). We have proved that $(\forall K \in \Omega)F(\overline{\gamma}K) = Y(s * \gamma)$ and underspill yields $(\forall^{\mathrm{st}} \gamma^1)(\exists^{\mathrm{st}} m)(\forall n \geq m)(\mathrm{st}(F(\overline{\gamma}n, M)))$ and hence (3.15).

To prove (3.16), assume the antecedent of the latter for standard $t$, and consider

$$\begin{aligned} F(t, M) &= G(Y, s * t, M) \\ &= Y\big(s * t * 0 * (\lambda n)G(Y, s * t * (n+1), M)\big) \\ &= Y\big(s * t * 0 * (\lambda n)F(t * (n+1), M)\big) && (4.6) \end{aligned}$$

which follows by the definitions of $F$ and $G$. However, the antecedent of (3.16) tells us that $F(t * \langle m \rangle, M)$ are standard for any standard $m$. Hence, the sequence

$$s * t * 0 * F(t * 1, M) * F(t * 2, M) * F(t * 3, M) * \dots \tag{4.7}$$

has a standard part by $\mathsf{STP}$, say $\gamma^1$, and $\mathsf{NPC}$ yields $F(t, M) = Y(\gamma)$, which is standard. Hence, we obtain (3.16), and $F(\langle\rangle, M) = G(Y, s, M)$ is standard by $\mathsf{EBI}$.

Now define the function $F(x, M)$ as follows:

$$F(x, M) := \begin{cases} 0 & G(Y, s * x, M) = G(Y, s * x, M+1) \\ M & \text{otherwise} \end{cases}, \tag{4.8}$$

where $Y^2 \in C$ and $s^0$ are standard again. Repeating the steps from the previous paragraph of the proof, we note that $F(\cdot, M)$ satisfies (3.15) and (3.16) for any

$M \in \Omega$. Hence, $\mathsf{EBI}$ yields that $F(\langle\rangle, M)$ is standard for any infinite $M$; As a consequence, we have $G(Y, s, M) = G(Y, s, M+1)$ by definition, for any infinite $M$. Hence, (4.3), the second half of Theorem 4.1, is proved. The proof of the first part is completely analogous. □

The Gandy-Hyland is unique as noted in [12, §6] and [15, §8.3.3]. We now prove a similar result, for which we require the following formula:

$$(\forall^{\rm st} Y^2 \in C, s^0)\big[\Gamma(Y^2, s^0) = Y\big(s * 0 * (\lambda n^0)\Gamma(Y, s*(n+1))\big)\big]. \qquad (\mathsf{GH}_{\rm st}(\Gamma))$$

The following corollary expresses that the standard and unique Gandy-Hyland functional equals its canonical approximations.

**Corollary 4.2.** *In $\mathsf{P} + \mathsf{NPC} + \mathsf{STP}$, the Gandy-Hyland functional exists and equals its canonical approximations, i.e. there is standard $\Gamma^3$ such that $\mathsf{GH}_{\rm st}(\Gamma)$ and*

$$(\forall^{\rm st} Y^2 \in C, s^0)(\forall N \in \Omega)(G(Y, s, N) = H(Y, s, N) = \Gamma(Y, s)). \qquad (\mathsf{CA}(\Gamma))$$

*Furthermore, the Gandy-Hyland functional is unique in that every functional $\Gamma$ such that $\mathsf{GH}_{\rm st}(\Gamma)$ satisfies $\mathsf{CA}(\Gamma)$.*

*Proof.* By (4.3), $G(Y, s, M)$ is $\Omega$-invariant and $\Omega$-$\mathsf{CA}$ and Remark 3.8 yield the standard part of $G(Y, s, M)$, say $\Gamma_0(Y, s)$. For standard $Y^2 \in C, s^0$ and $M \in \Omega$,

$$\begin{aligned}\Gamma_0(Y, s) = G(Y, s, M) &= Y\big(s * 0 * G(Y, s*1, M) * G(Y, s*2, M) * \dots\big) \qquad (4.9)\\ &= Y\big(s * 0 * \Gamma_0(Y, s*1) * \Gamma_0(Y, s*2) * \dots\big),\end{aligned}$$

where we used $\mathsf{NPC}$ in the final step. Hence, the standard part $\Gamma_0(\cdot)$ of $G(\cdot, M)$ as provided by $\Omega$-$\mathsf{CA}$ is indeed the Gandy-Hyland functional as $\mathsf{GH}_{\rm st}(\Gamma_0)$ follows from $\mathsf{CA}(\Gamma)$. To prove the uniqueness as in the corollary, suppose there is another $\Gamma_1$ such that $\mathsf{GH}_{\rm st}(\Gamma_1)$ and define $F(x, M)$ as in (4.8), but with $G(Y, s*x, M+1)$ replaced by $\Gamma_1(Y, s*x)$. Now proceed as in the proof of the theorem to establish that $\Gamma_0(Y, s) = G(Y, s, M) = \Gamma_1(Y, s)$ for standard $Y^2 \in C, s^0$ and $M \in \Omega$. □

As noted above, the Gandy-Hyland functional is not computable (in the sense of Kleene's S1-S9) in terms of the fan functional over the total continuous functionals. The following corollary expresses that the Gandy-Hyland functional as in $\mathsf{GH}(\Gamma)$ may be computed (via a term in Gödel's $\mathsf{T}$) from the special fan functional (See Corollary 3.3) and a modulus-of-continuity functional.

$$(\forall Y^2 \in C, s^0)\big[\Gamma(Y^2, s^0) = Y\big(s * 0 * (\lambda n^0)\Gamma(Y, s*(n+1))\big)\big]. \qquad (\mathsf{GH}(\Gamma))$$

$$(\forall Y^2 \in C, f^1, g^1)(\overline{f}\Psi(Y, f) = \overline{g}\Psi(Y, f) \to Y(f) = Y(g)). \qquad (\mathsf{MPC}(\Psi))$$

Variations of the following relative computability result are discussed below.

**Corollary 4.3** (Term extraction I)**.** *From the proof in $\mathsf{P}$ of*

$$\mathsf{NPC} + \mathsf{STP} \to (\forall \Gamma^3)\big[\mathsf{GH}_{\rm st}(\Gamma) \to \mathsf{CA}(\Gamma)\big], \qquad (4.10)$$

*a term $t^4$ can be extracted such that $\mathsf{E}\text{-}\mathsf{PA}^{\omega*}$ proves for all $\Theta, \Psi, \Gamma$ that*

$$\big[\mathsf{GH}(\Gamma) \wedge \mathsf{SCF}(\Theta) \wedge \mathsf{MPC}(\Psi)\big] \to (\forall Y \in C, s)\big(G(Y, s, t(Y, s, \Theta, \Psi) = \Gamma(Y, s)\big) \quad (4.11)$$

*i.e. $G(Y, s, t(Y, s, \Theta, \Psi))$ is the Gandy-Hyland functional expressed in terms of a modulus-of-continuity functional, and the special fan functional.*

*Proof.* The following formula is provable in $\mathsf{P}$ by Corollary 4.2:

$$(\forall \Gamma^3)\big[[\mathsf{GH}_{\mathrm{st}}(\Gamma) \wedge \mathsf{STP} \wedge \mathsf{NPC}] \to \mathsf{CA}(\Gamma)\big]. \tag{4.12}$$

We shall bring all the components in normal form and apply Corollary 2.4 to obtain the term $t$ from the theorem. First of all, resolving '$\approx_1$' in $\mathsf{NPC}$ implies

$$(\forall^{\mathrm{st}} Y^2 \in C, f^1)(\forall g^1)(\exists^{\mathrm{st}} N)(\overline{f}N =_0 \overline{g}N \to Y(f) =_0 Y(g)),$$

and applying idealisation $\mathsf{I}$, we obtain

$$(\forall^{\mathrm{st}} Y^2 \in C, f^1)(\exists^{\mathrm{st}} N)\big[(\forall g^1)(\overline{f}N =_0 \overline{g}N \to Y(f) =_0 Y(g))\big], \tag{4.13}$$

and let $A(Y, f, N)$ be the formula in square brackets. Secondly, let $(\forall^{\mathrm{st}} g^2)(\exists^{\mathrm{st}} w^{1^*})B(g, w)$ be the normal form of $\mathsf{STP}$ formulated in the proof of Corollary 3.3. Thirdly, Corollary 4.2 combined with underspill implies that for all $\Gamma$ such that $\mathsf{GH}_{\mathrm{st}}(\Gamma)$ we have

$$(\forall^{\mathrm{st}} Y^2 \in C, s^0)(\exists^{\mathrm{st}} K)\big[(\forall N \geq K)(G(Y, s, N) =_0 \Gamma(Y, s))\big],$$

and let $C(Y, s, K, \Gamma)$ be the formula in square brackets. Thus, the proof of (4.12) gives rise to a proof of the statement that for all $\Gamma$ and all *standard* $\Theta, \Psi$, we have

$$\begin{aligned}\big[(\forall^{\mathrm{st}} Y^2 \in C, f^1)A(Y, f, \Psi(Y, f) \wedge (\forall^{\mathrm{st}} W^2)B(W, \Theta(W)) \wedge \mathsf{GH}_{\mathrm{st}}(\Gamma)\big]& \\ \to (\forall^{\mathrm{st}} Z^2 \in C, s)(\exists^{\mathrm{st}} N)C(Z, s, N, \Gamma),&\end{aligned} \tag{4.14}$$

and strengthening the antecedent of (4.14) to internal formulas:

$$\begin{aligned}&(\forall^{\mathrm{st}} \Theta, \Psi, Z \in C, s)(\forall \Gamma)(\exists^{\mathrm{st}} N^0) \\ &\big[[(\forall Y \in C, f)A(Y, f, \Psi(Y, f) \wedge (\forall W^2)B(W, \Theta(W)) \wedge \mathsf{GH}(\Gamma)] \to C(Z, s, N, \Gamma)\big].\end{aligned} \tag{4.15}$$

Apply idealisation $\mathsf{I}$ to pull the existential quantifier through '$(\forall \Gamma)$' as follows:

$$\begin{aligned}&(\forall^{\mathrm{st}} \Theta, \Psi, Z \in C, s)(\exists^{\mathrm{st}} N^0) \\ &(\forall \Gamma)\big[[(\forall Y \in C, f)A(Y, f, \Psi(Y, f) \wedge (\forall W^2)B(W, \Theta(W)) \wedge \mathsf{GH}(\Gamma)] \to C(Z, s, N, \Gamma)\big].\end{aligned}$$

Apply Corollary 2.4 to obtain[8] a term $t$ such that $\mathsf{E\text{-}PA}^{\omega*}$ proves for $\Theta, \Psi, Z \in C, s, \Gamma$

$$((\forall Y \in C, f)A(Y, f, \Psi(Y, f)) \wedge (\forall W^2)B(W, \Theta(W)) \wedge \mathsf{GH}(\Gamma)) \to C(Z, s, t(\Theta, \Psi, Z, s), \Gamma).$$

Clearly, $(\forall Y \in C, f)A(Y, f, \Psi(Y, f))$ expresses that $\Psi$ is a modulus-of-continuity functional, and $(\forall W^2)B(W, \Theta(W))$ expresses that $\Theta$ is the special fan functional as in $\mathsf{SCF}(\Theta)$. Bringing the $Z$ and $s$ quantifiers into the consequent, we get (4.11). $\square$

We now discuss possible strengthenings of the above results.

**Remark 4.4** (Similar results)**.** It is an interesting question if the condition '$Y^2 \in C$' in the previous results can be weakened. First of all, we *cannot* drop this condition: As shown in the proof of Corollary 4.3, $\mathsf{NPC}$ gives rise to a modulus-of-continuity functional $\Psi$ as in $\mathsf{MPC}(\Psi)$. Intuitively speaking, $\Psi$ is rather discontinuous and from the existence of the latter for *all* type two functionals (i.e. without the restriction '$Y^2 \in C$' in $\mathsf{MPC}(\Psi)$), one constructs a *discontinuous* type two functional, yielding a contradiction (See [10] and [3, Theorem 19.1]). Secondly, going through the proofs in this section, it is becomes clear that '$Y^2 \in C$' can be replaced everywhere by any *internal* formula $D(Y^2)$, as long as the latter formula blocks the aforementioned contradiction in the same way as '$Y^2 \in C$' does.

[8] Corollary 2.4 provides a finite sequence of possible witnesses, of which $t$ is the maximum.

In light of the previous remark and [13, Prop. 3.7], the existence of a modulus-of-continuity functional implies the Halting problem. In Sections 4.3 and 4.4, we obtain relative computability results similar to (4.11) with weaker antecedents. We now discuss how the *consequent* of the above results can be strengthened.

**Remark 4.5** (Similar results II)**.** It is a natural question if the consequent of (4.11) is the best possible result. A careful study of the proofs of Theorem 4.1 and Corollary 4.2 reveals the existence of standard $\Gamma^3$ such that $\mathsf{GH}_{\rm st}(\Gamma)$ and

$$(\forall^{\rm st} Y^2 \in C, \alpha^1)(\forall N^0, M^0 \in \Omega) \qquad (\forall s^0)(\overline{\alpha}M =_0 \overline{s * 00\dots}M \to \Gamma(Y, s) =_0 G(Y, s, N)). \tag{4.16}$$

Indeed, if $s^0$ is standard, then the associated instance of (4.16) follows from the above results. If $s^0$ is nonstandard, it has $\alpha$ as a standard part and (4.16) follows by nonstandard continuity as in $\mathsf{NPC}$. Applying term extraction to a variation of (4.10) involving (4.16), one obtains a term $u$ which computes the Gandy-Hyland functional 'more uniformly' than the term $t$ in (4.11), in that $u$ provides *one* stopping condition for every initial segment $s^0$ of the sequence $\alpha^1$. We shall obtain an equivalence between (4.16) and $\mathsf{NPC}$ in Section 4.5.

In conclusion, we have proved in Theorem 4.1 that the functional $H(\cdot, M)$ and $\Gamma(\cdot)$ are equal for standard inputs and nonstandard $M^0$. From this proof, we have extracted a term from Gödel's $\mathsf{T}$ which computes the $\Gamma$-functional as a function of the special fan functional and a modulus-of-continuity functional. While these relative computability results are not necessarily deep or surprising, our methodology constitutes the true surprise: That from the rather ineffective proof of Theorem 4.1 *involving Nonstandard Analysis*, the term $t$ as in Corollary 4.3 may be extracted.

4.2. **From relative computability to Nonstandard Analysis.** In the previous section, we showed how to extract relative computability results like (4.11) from corresponding nonstandard statements like (4.10). Now, it is a natural 'Reverse Mathematics style' question whether it is possible to re-obtain the nonstandard implication from (a variation of) the associated relative computability result.

Another natural question is whether we can obtain a version of (4.11) with weaker assumptions; Indeed, to compute $\Gamma(Y, s)$ it should -intuitively speaking-suffice to have a functional which (only) behaves like the special fan and modulus-of-continuity functional for $Y$ (and functionals explicitly defined from the latter).

To answer these two questions, we define the *Hebrandisation* of (4.11) as follows. Let $\mathsf{SCF}(\Theta, g)$ be $\mathsf{SCF}(\Theta)$ with the leading quantifier involving $g$ dropped. Let $\mathsf{MPC}(\Psi, Y, f)$ be $\mathsf{MPC}(\Psi)$ with the leading quantifier involving $Y$ and $f$ dropped. Let $\mathsf{GH}(\Gamma, Y, s)$ be $\mathsf{GH}(\Psi)$ with the leading quantifier involving $Y$ and $s$ dropped.

**Definition 4.6** (Herbandisation)**.** The *Herbrandisation* $\mathsf{HER}(i, o)$ of (4.12) is the statement that for all $\Xi = (\Theta, \Psi)$ and all $\Gamma^3, Y^2 \in C, s^0$

$$\big[\big(\forall (Z^2 \in C, t^0) \in i(Y, s, \Xi)(1)\big)\big)\mathsf{GH}(\Gamma, Z, t) \wedge (\forall W^2 \in i(Y, s, \Xi)(2))\mathsf{SCF}(\Theta, W)$$
$$\wedge \big(\forall (V^2 \in C, f^1) \in i(Y, s, \Xi)(3)\big)\mathsf{MPC}(\Psi, V, f)\big] \to (\forall M \geq o(Y, s, \Xi))\big(G(Y, s, M) = \Gamma(Y, s)\big).$$

Intuitively speaking, the Herbrandisation $\mathsf{HER}(i, o)$ expresses that to compute $\Gamma(Y, s)$ via its canonical approximation involving the term $o$, it suffices that the special fan and modulus-of-continuity functionals satisfy their usual definition *on*

*the restriction of their domains provided by* $i$. By the following corollary, the nonstandard version (4.12) has the same computational content as its Herbrandisation.

**Theorem 4.7.** *From the proof of* (4.12) *in* $\mathsf{P}$*, terms* $i^4, o^4$ *can be extracted such that* $\textsf{E-PA}^{\omega*}$ *proves* $\mathsf{HER}(i,o)$*. Furthermore, if there are terms* $i, o$ *such that the system* $\textsf{E-PA}^{\omega*}$ *proves* $\mathsf{HER}(i,o)$*, then* $\mathsf{P}$ *proves* (4.12)*.*

*Proof.* For the first part of the corollary, consider the proof of Corollary 4.3 and obtain the following *non-weakened* form of (4.15):

$$(\forall^{\rm st}\Theta, \Psi, Z \in C, s)(\forall \Gamma^3)(\exists^{\rm st} N^0, Y \in C, f, g, V, t)\big[[A(Y, f, \Psi(Y, f)) \wedge \mathsf{SCF}(\Theta, g) \wedge \mathsf{GH}(\Gamma, V, t)] \rightarrow C(Z, s, N, \Gamma)\big]. \tag{4.17}$$

Now apply idealisation $\mathsf{I}$ to obtain

$$(\forall^{\rm st}\Theta, \Psi, Z \in C, s)(\exists^{\rm st} W)(\forall \Gamma^3)(\exists (N^0, Y \in C, f, g, V, t) \in W) \big[[A(Y, f, \Psi(Y, f)) \wedge \mathsf{SCF}(\Theta, g) \wedge \mathsf{GH}(\Gamma, V, t)] \rightarrow C(Z, s, N, \Gamma)\big], \tag{4.18}$$

and apply Corollary 2.4 to obtain a term $w$ such that $\textsf{E-PA}^{\omega*}$ proves

$$(\forall \Theta, \Psi, Z \in C, s)(\exists W \in w(\Theta, \Psi, Z, s))(\forall \Gamma^3)(\exists (N^0, Y \in C, f, g, V, t) \in W) \big[[A(Y, f, \Psi(Y, f)) \wedge \mathsf{SCF}(\Theta, g) \wedge \mathsf{GH}(\Gamma, V, t)] \rightarrow C(Z, s, N, \Gamma)\big].$$

Define the term $o$ as follows: $o(\Theta, \Psi, Z, s, )$ is the maximum of the components of $w(\Theta, \Psi, Z, s)$ pertaining to $N$; Similarly, define the term $i(\Theta, \Psi, Z, s)(j)$ for $j = 1$ (resp. $j = 2$ and $j = 3$) to be the finite sequence of all components of $w$ pertaining to the variables $Y, f$ (resp. $g$ and $V, t$). Thus, we obtain $\mathsf{HER}(i, o)$ as defined above.

For the second part, if there are terms $i, o$ such that $\textsf{E-PA}^{\omega*} \vdash \mathsf{HER}(i, o)$, then $\mathsf{P} \vdash \big[\mathsf{HER}(i, o) \wedge \mathrm{st}(i) \wedge \mathrm{st}(o)\big]$ by the second standardness axiom from Definition 2.1. Thus, for *standard* $\Xi = (\Theta, \Psi)$ and standard $Y^2 \in C, s^0$, the terms $i(Y, s, \Xi)$ and $o(Y, s, \Xi)$ are standard (by the third standardness axiom from Definition 2.1), and $\mathsf{HER}(i, o)$ implies the following weakening (for any $\Gamma^3$):

$$\big[(\forall^{\rm st} Z^2 \in C, t^0)\mathsf{GH}(\Gamma, Z, t) \wedge (\forall^{\rm st} W^2)\mathsf{SCF}(\Theta, W) \wedge (\forall^{\rm st} V^2 \in C, f^1)\mathsf{MPC}(\Psi, V, f)\big] \rightarrow (\forall M \in \Omega)\big(G(Y, s, M) = \Gamma(Y, s)\big), \tag{4.19}$$

Applying $\mathsf{HAC}_{\mathsf{int}}$ to (4.13), $\mathsf{NPC}$ implies $(\exists^{\rm st}\Psi)(\forall^{\rm st} Y \in C, f)\mathsf{MPC}(\Psi, Y, f)$. Similarly, $\mathsf{STP}$ implies $(\exists^{\rm st}\Omega^3)(\forall^{\rm st} Y^2)\mathsf{SCF}(\Theta, Y)$ by the proof of Corollary 3.3. Hence, $\mathsf{NPC}+\mathsf{STP}$ implies the second and third conjunct of the antecedent of (4.19), which yields (4.12), and we are done. ☐

In this section, we have proved that from a proof of (4.12), terms $i, o$ from Gödel's $\mathsf{T}$ can be extracted which satisfy the so-called Herbrandisation of (4.12), i.e. $o$ computes $\Gamma(Y, s)$ as a function of *approximations enforced by* $i$ of the special fan functional and a modulus-of-continuity functional. Furthermore, the nonstandard version (4.12) implies its Herbrandisation, i.e. *they have the same computational content* in the sense of Theorem 4.7.

In conclusion, the correspondence exhibited in Theorem 4.7 establishes a direct two-way connection between the field Computability (in particular theoretical computer science) and the field Nonstandard Analysis. Indeed, while the relative computability result $\mathsf{HER}(i, o)$ could arguably still be passed off as (theoretical) computer science, experience bears out that the nonstandard version (4.10) does not count as such among computer scientists.

4.3. **From Nonstandard Analysis to relative computability II.** In this section, we obtain a relative computability result involving the Gandy-Hyland functional, but not mentioning the modulus-of-continuity functional. To this end, we shall establish that the proofs of Theorem 4.1 and Corollary 4.2 also go through for other principles appearing in the context of bar recursion.

First of all, consider the 'full' fan functional as follows:

$$(\forall Y^2 \in C, h^1)(\forall f^1, g^1 \leq_1 h)(\overline{f}\Omega(Y,h) =_0 \overline{g}\Omega(Y,h) \to Y(f) =_0 Y(g)), \quad (\mathsf{FFF}(\Omega))$$

and the related nonstandard continuity principle:

$$(\forall^{\text{st}} Y^2 \in C, h^1)(\forall f^1, g^1 \leq_1 h)(f \approx_1 g \to Y(f) =_0 Y(g)). \quad (\mathsf{FFF_{ns}})$$

Secondly, the *weak continuity functional* is defined as follows:

**Principle 4.8** ($\mathsf{PWC}$)**.** *There is a functional $\Upsilon^3$ such that*

$$(\forall Y^2 \in C, f^1, g^1)\big[\overline{f}\Upsilon(Y,f) = \overline{g}\Upsilon(Y,f) \to Y(g) \leq \Upsilon(Y,f)\big], \quad (\mathsf{PWC}(\Upsilon))$$

*and $\Upsilon(Y,f)$ is the least number with this property.*

The nonstandard principle associated to $\mathsf{PWC}$ is the following (See also (4.20)):

$$(\forall^{\text{st}} Y^2 \in C, f^1)(\forall g^1)\big[f \approx_1 g \to \text{st}(Y(g))\big]. \quad (\mathsf{PWC_{ns}})$$

We refer to [6, §2.5 and Lemma 5.4] for more details on these functionals and their relation to bar recursion.

**Theorem 4.9.** *The proofs from Theorem 4.1 and Corollary 4.2 in* $\mathsf{P} + \mathsf{NPC} + \mathsf{STP}$ *go through for* $\mathsf{NPC}$ *replaced by* $\mathsf{FFF_{ns}}$ *and* $\mathsf{PWC_{ns}}$*.*

*Proof.* It is straightforward to prove that $\mathsf{PWC_{ns}}$ has the following normal form:

$$(\forall^{\text{st}} Y^2 \in C, f^1)(\exists^{\text{st}} n)(\forall g^1)\big[\overline{f}n = \overline{g}n \to Y(g) \leq n\big], \tag{4.20}$$

In the proof of Theorem 4.1 in Section 4.1, every instance of nonstandard (pointwise) continuity $\mathsf{NPC}$ can be replaced by combining nonstandard continuity via $\mathsf{FFF_{ns}}$ and nonstandard weak continuity via $\mathsf{PWC_{ns}}$.

For instance, to obtain that (4.4) is standard, apply the weak continuity principle $\mathsf{PWC_{ns}}$ to $Y \in C$ and $s * \gamma$. Similarly, the sequence in (4.7) has a standard part by $\mathsf{STP}$. Applying the weak continuity principle $\mathsf{PWC_{ns}}$ to this standard part, (4.6) turns out to be a standard number.

The second part of the proof involving (4.7) is similar: One uses the weak continuity principle $\mathsf{PWC_{ns}}$ and $\mathsf{STP}$ to obtain a standard upper bound for all sequences involved, and $\mathsf{FFF_{ns}}$ provides the required continuity (using the aforementioned upper bound as $h^1$ in $\mathsf{FFF_{ns}}$).

For instance, for standard $\gamma^1, Y^2 \in C, s^0$ and $N \in \Omega$, the sequences

$$\beta_0 := s * \overline{\gamma}N * 0 * (\lambda n)G(Y, s * \overline{\gamma}N * (n+1), M)$$

$$\beta_1 := s * \overline{\gamma}N * 0 * (\lambda n)G(Y, s * \overline{\gamma}N * (n+1), M+1)$$

satisfy $\beta_0 \approx_1 \beta_1$ and $\beta_0, \beta_1 \leq_1 h$, where $h(i) := \max\{\Upsilon(Y, s*\gamma), \gamma(i), \overline{s * 00\ldots}(i)\}$ and where $\Upsilon$ results from applying $\mathsf{HAC_{int}}$ to (4.20) and defining $\Upsilon(Y,f)$ as the maximum of the output sequence. Note that $h^1$ is standard and apply $\mathsf{FFF_{ns}}$ for $Y^2 \in C$ and $\beta_1, \beta_0 \leq_1 h$ to obtain:

$$G(Y, s * \overline{\gamma}N, M) = Y(\beta_0) = Y(\beta_1) = G(Y, s * x, M+1),$$

implying that $F(\overline{\gamma}N, M) = 0$ and $(\forall^{\text{st}}\gamma)(\forall N \in \Omega)\text{st}(F(\overline{\gamma}N, M))$. Underspill now yields (3.15). The remaining part of the proof, namely (3.16), is now analogous. □

**Corollary 4.10** (Term extraction II)**.** *From the modified proofs of Theorem 4.1 and Corollary 4.2 inside* $\mathsf{P} + \mathsf{STP} + \mathsf{FFF}_{\mathsf{ns}} + \mathsf{PWC}_{\mathsf{ns}}$*, a term* $t^4$ *can be extracted such that* $\mathsf{E\text{-}PA}^{\omega*}$ *proves for all* $\Omega^3, \Upsilon^3, \Gamma^3, \Theta$ *that*

$$\mathsf{GH}(\Gamma) \wedge \mathsf{SCF}(\Theta) \wedge \mathsf{FFF}(\Omega) \wedge \mathsf{PWC}(\Upsilon) \\ \to (\forall Y^2 \in C, s^0)\big(G(Y, s, t(Y, s, \Omega, \Upsilon, \Theta)) = \Gamma(Y, s)\big),$$

*i.e.* $G(Y, s, t(Y, s, \Omega, \Upsilon, \Theta))$ *is the Gandy-Hyland functional expressed in terms of the 'full' fan functional, the special fan functional, and a weak continuity functional.*

*Proof.* Analogous to Corollary 4.3. A normal form of $\mathsf{PWC}_{\mathsf{ns}}$ is (4.20). □

Following Definition 4.6, it is easy to define the Herbrandisation of (4.12) for $\mathsf{NPC}$ replaced by $\mathsf{FFF}_{\mathsf{ns}}$ and $\mathsf{PWC}_{\mathsf{ns}}$, and obtain a result similar to Theorem 4.7.

4.4. **From Nonstandard Analysis to relative computability III.** In this section, we obtain a 'pointwise' relative computability result involving the Gandy-Hyland functional, which additionally only mentions the special fan functional.

To this end, we shall establish that the proofs of Theorem 4.1 and Corollary 4.2 also go through 'in a pointwise fashion', to be understood in the sense of Theorem 4.11. Recall from Section 4.2 the formula $\mathsf{GH}(\Gamma, Y, s)$ and define $\mathsf{NPC}(Y)$ as $\mathsf{NPC}$ with the leading universal quantifier involving $Y$ dropped.

**Theorem 4.11.** *The system* $\mathsf{P} + \mathsf{STP}$ *proves that*

$$(\forall^{\text{st}} Y, s)(\forall \Gamma^3)\Big[\big[\mathsf{NPC}(Y) \wedge \mathsf{GH}(\Gamma, Y, s)\big] \to (\forall N \in \Omega)(\Gamma(Y, s) = G(Y, s, N))\Big]. \tag{4.21}$$

*Proof.* Fix $\Gamma^3$ and standard $Y^2, s^0$ as in the antecedent of (4.21). In exactly the same was as in the first part of the proof of Theorem 4.1, one proves that $G(Y, s, N)$ is standard for any $N \in \Omega$. Indeed, in the aforementioned proof part, $\mathsf{NPC}$ is only applied to the functional $Y^2$, i.e. $\mathsf{NPC}(Y)$ is sufficient to prove $\text{st}(G(Y, s, N))$.

Analogous to (4.8), fix $N \in \Omega$ and define $F(x, N)$ as follows:

$$F(x, N) := \begin{cases} 0 & G(Y, s * x, N) =_0 \Gamma(Y, s * x) \\ N & \text{otherwise} \end{cases}. \tag{4.22}$$

We shall use $\mathsf{EBI}$ to prove that $\text{st}(F(\langle\rangle, N))$, finishing the proof. To prove (3.15), note that for $M \in \Omega$ and standard $\alpha^1$, we have $G(Y, \overline{\alpha}M, N) = Y(\alpha) = \Gamma(Y, \overline{\alpha}M)$, by the nonstandard continuity of $Y^2$ and the fact that $(\overline{\alpha}M * \gamma) \approx_1 \alpha$ for any $\gamma^1$. As in the proof of Theorem 4.1, one obtains (3.15) using underspill.

To prove (3.16), fix standard $t^0$ and assume $\text{st}(F(t * \langle x\rangle))$ for all standard $x^0$. By (4.22), we have $G(Y, s*t*\langle x\rangle, N) = \Gamma(Y, s*t*\langle x\rangle)$ for any standard $x^0$. By the previous part of the proof and $\mathsf{STP}$, the sequence $s*t*0*(\lambda n)G(Y, s*t*(n+1)), N)$ has a standard part, say $\gamma^1$. By assumption, we have

$$s * t * 0 * (\lambda n)G(Y, s * t * (n + 1)), N) \approx_1 \gamma \approx_1 s * t * 0 * (\lambda m)\Gamma(Y, s * t * (m + 1)),$$

and the nonstandard continuity of $Y$ yields:

$$\begin{aligned} G(Y, s * t, N) &= Y(s * t * 0 * (\lambda n)G(Y, s * t * (n + 1)), N) \\ &= Y(s * t * 0 * (\lambda m)\Gamma(Y, s * t * (m + 1))) = \Gamma(Y, s * t), \end{aligned} \tag{4.23}$$

which implies that $\mathrm{st}(F(s * t, N))$, and (3.16) follows. □

Let $\mathsf{PCM}(Y^2, H^2)$ be the expression that $H$ is a modulus of continuity for $Y$, i.e. we have (4.1) and $N^0$ is additionally given by $H(f)$.

**Corollary 4.12** (Term extraction III)**.** *From the proof of* (4.21) *in* $\mathsf{P} + \mathsf{STP}$*, a term $t$ can be extracted such that* $\mathsf{E\text{-}PA}^{\omega*}$ *proves for all* $Y, s, g, \Gamma, \Theta$ *that*

$$\big[\mathsf{PCM}(Y, g) \wedge \mathsf{SCF}(\Theta) \wedge \mathsf{GH}(\Gamma, Y, s)\big] \rightarrow (\forall N \geq t(Y, s, g, \Theta))(\Gamma(Y, s) = G(Y, s, N)),$$

*i.e. the value of the Gandy-Hyland functional can be computed from a modulus of continuity of the input functional, and the* (*special*) *fan functional.*

*Proof.* We sketch how one obtains a normal form for $\mathsf{STP} \rightarrow$ (4.21); Applying Corollary 2.4 then finishes the proof. The normal form of $\mathsf{STP}$ has been studied in the proof Corollary 4.3. The normal form of $\mathsf{NPC}(Y)$, obtained in the same way as the normal form of $\mathsf{NPC}$ in the proof of Corollary 4.3, is

$$(\forall^{\mathrm{st}} f^1)(\exists^{\mathrm{st}} N)\big[(\forall g^1)(\overline{f}N =_0 \overline{g}N \rightarrow Y(f) =_0 Y(g))\big],$$

and applying $\mathsf{HAC}_{\mathsf{int}}$, one sees how the modulus of continuity of $Y$ comes about. □

Following Definition 4.6, it is easy to define the Herbrandisation of (4.21) and obtain a result similar to Theorem 4.7. In particular, this Hebrandisation tells us on which part of Baire space the functional $Y$ should be continuous (with modulus $H$) to guarantee that $\Gamma$ and $G$ coincide at $Y$.

Thus, we have obtained a 'pointwise' relative computability result involving the Gandy-Hyland functional which only mentions the special fan functional. In particular, the term $t$ from Corollary 4.12 allows us to compute the value of the Gandy-Hyland functional in terms of the (special) fan functional for any functional $Y$ with a modulus of (pointwise) continuity. It should be noted that the statement *every continuous functional on Baire space has a modulus of pointwise continuity*, is rather weak by [14, Prop. 4.4 and 4.8].

We finish this section with a remark on how the relative computability results in this paper may be used. We thank Dag Normann for his advice in these matters.

**Remark 4.13** (Known associates)**.** As noted above, the fan and Gandy-Hyland functionals are not computable over the total continuous functionals in the sense of Kleene's S1-S9 schemas. Nonetheless, these functionals do have a computable *representation* in the form of a type one *recursive associate* (See [17, §4]). In the case of the fan functional, this associate has even been implemented in Haskell ([8]).

Furthermore, all terms from Gödel's $\mathsf{T}$ have canonical interpretations in the type structure of the Kleene-Kreisel countable functionals (See [17, §2] for the latter) and this interpretation provides canonical associates. Thus, in Kleene's *second model* (See e.g. [3, §7.4, p. 132]), for a term $t^4$ of Gödel's $\mathsf{T}$ and $\Omega$ the fan functional, $t(\Omega)$ can be computed by evaluating the associate for $t$ on the associate for $\Omega$. Similarly, every functional $Y^2$ with a modulus of (pointwise) continuity $H^2$ has an associate defined explicitly in terms of $Y$ and $H$ (See [14, §4]).

Thus, the *term extraction* result in Corollary 4.12 express the associate of the objects being approximated in terms of the associates of the input objects.

4.5. **From Nonstandard Analysis to relative computability IV.** In this section, we show that the approximation of the Gandy-Hyland functional as in (4.16) is *equivalent* to $\mathsf{NPC}$, and derive the associated relative computability result in which a term from Gödel's $\mathsf{T}$ expresses a modulus-of-continuity functional in terms of an 'approximation' functional for the Gandy-Hyland functional.

**Principle 4.14** ($\mathsf{GHS}$). *There is standard* $\Gamma^3$ *such that* $\mathsf{GH}_{\mathsf{st}}(\Gamma)$ *and*

$$(\forall^{\mathrm{st}} Y^2 \in C, \alpha^1)(\forall N^0, M^0 \in \Omega)(\forall s^0)(\overline{\alpha}M =_0 \overline{s * 00\dots}M \to \Gamma(Y, s) =_0 H(Y, s, N)).$$

**Theorem 4.15.** *In* $\mathsf{P}$*, we have* $\mathsf{GHS} \to \mathsf{NPC}$*.*

*Proof.* In two words, to obtain $\mathsf{NPC}$ from $\mathsf{GHS}$, one computes for $Y^2 \in C$ the numbers $Y(\alpha)$ and $Y(\beta)$ using $\mathsf{GHS}$ and notes that they are identical if $\alpha \approx_1 \beta$ for standard $\alpha^1$ and any $\beta^1$. In more detail: Applying underspill to $\mathsf{GHS}$, we get

$$(\forall^{\mathrm{st}} Y^2 \in C, \alpha^1)(\exists^{\mathrm{st}} K^0)(\forall N^0, M^0 \geq K)(\forall s^0)(\overline{\alpha}M =_0 \overline{s * 00\dots}M \to \Gamma(Y, s) =_0 H(Y, s, N)).$$

Applying $\mathsf{HAC}_{\mathsf{int}}$ to the previous formula yields a standard functional $\Xi^3$ such that

$$\begin{aligned}(\forall^{\mathrm{st}} Y^2 \in C, &\alpha^1)(\forall N^0, M^0 \geq \Xi(Y, \alpha))\\ &(\forall s^0)(\overline{\alpha}M =_0 \overline{s * 00\dots}M \to \Gamma(Y, s) =_0 H(Y, s, N)).\end{aligned} \tag{4.24}$$

Now fix standard $Y^2 \in C$ and standard $\alpha^1$, and any $\beta^1$ such that $\alpha \approx_1 \beta$. Since $Y^2 \in C$, there are numbers $N_0^0, M_0^0$ such that

$$(\forall \gamma^1)(\overline{\gamma}M_0 =_0 \overline{\beta}M_0 \to Y(\gamma) =_0 Y(\beta)). \tag{4.25}$$

$$(\forall \gamma^1)(\overline{\gamma}N_0 =_0 \overline{\alpha}N_0 \to Y(\gamma) =_0 Y(\alpha)). \tag{4.26}$$

If $M_0$ or $N_0$ is standard, we have $Y(\alpha) = Y(\beta)$, and we are done. Otherwise, we have $Y(\overline{\alpha}N_0 * 00\dots) = Y(\alpha)$ and $Y(\overline{\beta}M_0 * 00\dots) = Y(\beta)$ by (4.25) and (4.26). The following equalities now follow from the above. Note that $\Xi(Y, \alpha)$ is standard.

$$\begin{aligned}
Y(\alpha) = Y(\overline{\alpha}N_0 * 00\dots) = H(Y, \overline{\alpha}N_0, N_0) &= \Gamma(Y, \overline{\alpha}N_0) && (4.27)\\
&= H(Y, \overline{\alpha}N_0, \Xi(Y, \alpha)) && (4.28)\\
&= Y(\overline{\alpha}\Xi(Y, \alpha) * 00\dots)\\
&= Y(\overline{\beta}\Xi(Y, \alpha) * 00\dots)\\
&= H(Y, \overline{\beta}M_0, \Xi(Y, \alpha))\\
&= \Gamma(Y, \overline{\beta}M_0)\\
&= H(Y, \overline{\beta}M_0, M_0)\\
&= Y(\overline{\beta}M_0 * 00\dots) = Y(\beta).
\end{aligned}$$

For instance, to conclude (4.28) from (4.27), one applies (4.24) for $s = \overline{\alpha}N_0$ and $N = \Xi(Y, \alpha)$ and $M = N_0$. The other equalities are proved similarly. $\square$

**Corollary 4.16.** *In* $\mathsf{P} + \mathsf{STP}$ *we have* $\mathsf{GHS} \leftrightarrow \mathsf{NPC}$*.*

Now define $\mathsf{GHS}(\Psi, \Gamma)$ as the following formula:

$$(\forall Y^2 \in C, \alpha^1, N, M \geq \Psi(Y, \alpha), s^0)\big(\overline{\alpha}M = \overline{s * 00\dots}M \to H(Y, s, N) =_0 \Gamma(Y, s)\big)\big),$$

which expresses that $\Psi$ witnesses the canonical approximation of $\Gamma$ via $H$ in a 'more uniform way' than is the case of $\mathsf{CA}(\Gamma)$.

**Corollary 4.17** (Term extraction)**.** *From the proof* $\mathsf{P}+\mathsf{GHS}\vdash\mathsf{NPC}$*, a term* $t^4$ *can be extracted such that* $\mathsf{E}\text{-}\mathsf{PA}^{\omega*}$ *proves for all* $\Gamma^3,\Psi^3$ *that*

$$\big[\mathsf{GH}(\Gamma)\wedge\mathsf{GHS}(\Psi,\Gamma)\big]\to\mathsf{MPC}(t(\Gamma,\Psi)).\tag{4.29}$$

*Proof.* Analogous to the proof of Corollary 4.3. □

An analogous result for weak continuity as in PWC is now easily obtained. Following Definition 4.6, it is easy to define the Herbrandisation of $\mathsf{GHS}\to\mathsf{NPC}$, and obtain a result as in Theorem 4.7.

In light of (4.29) and (4.11), there are terms from Gödel's T expressing the Gandy-Hyland functional as a modulus-of-continuity functional and vice versa. As it turns out, there is also a rather recent model-theoretic characterisation of the Gandy-Hyland functional, namely [15, Theorem 9.5.4, p. 460], which states that the totality of the Gandy-Hyland functional in a (computationally closed) model is equivalent to that model satisfying arithmetical comprehension.

Consequently, it is a natural question (due to Dag Normann) whether our results are related to the aforementioned model-theoretic result (involving partial functionals). We first discuss the connection between arithmetical comprehension and the existence of a modulus-of-continuity functional *without* Nonstandard Analysis.

**Remark 4.18.** As it turns out, the existence of a modulus-of-continuity functional as in $(\exists\Psi)\mathsf{MPC}(\Psi)$ is equivalent to arithmetical comprehension $(\mu^2)$ as follows:

$$(\exists\mu^2)\big[(\forall f^1)\big)(\exists n^0)f(n)=0\to f(\mu(f))=0\big)\big].\tag{$\mu^2$}$$

Indeed, for the forward implication, consider [10, p. 156] where a discontinuous functional is defined from a modulus-of-continuity functional. The existence of the former is proved to yield $(\mu^2)$ in [13, Prop. 3.7 and 3.9]. On the other hand, following the proof of [14, Prop. 4.7], one can (explicitly) define a modulus-of-continuity functional in terms of arithmetical comprehension $(\mu^2)$.

In light of the previous remark, the term $t$ in the consequent of (4.29) can be modified to obtain $(\mu^2)$, modulo the use of the quantifier-free axiom of choice as in the proofs of [13, Prop. 3.7 and 3.9]; Similar modifications for (4.11) are possible.

While the aforementioned results regarding arithmetical comprehension and the $\Gamma$-functional bear *some* resemblance to [15, Theorem 9.5.4, p. 460], they are not really satisfactory. On the other hand, how *would* one express totality in a system like P *where all functionals are total anyway*? We discuss these matters in the next sections where we also improve upon the results of this section.

4.6. **From Nonstandard Analysis to relative computability V.** In this section, we study the equivalences between NPC, the $\Gamma$-functional approximation as in Corollary 4.2, and the following fragment of Nelson's axiom *Transfer*:

$$(\forall^{\mathrm{st}}f^1)\big[(\forall^{\mathrm{st}}n)f(n)=0\to(\forall m)f(m)=0\big]\tag{$\Pi_1^0$-TRANS}$$

From these equivalences, we obtain relative computability results for arithmetical comprehension, the $\Gamma$-funtional, and a modulus-of-continuity functional. As it turns out, $\Pi_1^0$-TRANS is the 'precursor' of $(\mu^2)$ in the same way NPC is for $(\exists\Psi)\mathsf{MPC}(\Psi)$.

In this section, we shall work with *associates* of continuous functionals, rather than the functionals themselves. We first consider the definition of associate from [14, Def. 4.3] and then motivate why this choice changes very little.

**Definition 4.19.** [Associate] The function $\gamma^1$ is an *associate* of $Y^2 \in C$ if:

(i) $(\forall \beta^1)(\exists k^0)\gamma(\overline{\beta}k) > 0$,
(ii) $(\forall \beta^1, k^0)(\gamma(\overline{\beta}k) > 0 \to Y(\beta) + 1 =_0 \gamma(\overline{\beta}k))$.

We assume an associate $\gamma^1$ to be a *neighbourhood function* (See [14, §4]), i.e.

$$(\forall \sigma^0, \tau^0, n^0)\big((\sigma \preceq \tau \wedge \gamma(\overline{\sigma}n) > 0) \to \gamma(\sigma) =_0 \gamma(\tau)\big). \tag{4.30}$$

We now argue why working with associates is natural (in our context). First of all, consider the following fragment of the axiom of choice.

**Definition 4.20** ($\mathsf{QF\text{-}AC}^{1,0}$)**.** *For internal and quantifier-free $\varphi_0$, we have*

$$(\forall x^1)(\exists y^0)\varphi_0(x, y) \to (\exists F^2)(\forall x^1)\varphi_0(x, F(x)). \tag{4.31}$$

Applying $\mathsf{QF\text{-}AC}^{1,0}$ to item (i) in Definition 4.19, the latter gives rise to a continuous functional $Y^2$ by putting $Y(\alpha) := \gamma(\overline{\alpha}F(\alpha)) - 1$. As it turns out, $\mathsf{QF\text{-}AC}^{1,0}$ is a very weak principle, as established in [13, §2].

Secondly, as noted above, the proof of [14, Prop. 4.4] contains an explicit definition for obtaining an associate from a functional and its modulus of pointwise continuity. Hence, in the presence of a modulus-of-continuity functional, working with associates rather than the continuous functionals themselves seems to amount to the same thing. Finally, the logical framework for *Reverse Mathematics* ([21]) is second-order arithmetic, and one is thence forced to work with associates to study e.g. continuous functionals on Baire or Cantor space.

In light of the previous observations, it seems acceptable to work with associates directly, in the context of this paper. Thus, let us denote by '$\gamma^1 \in C$' the first item of Definition 4.19 plus the requirement on neighbourhood functions (4.30). We shall denote the 'value of $\gamma^1 \in C$ at $\alpha^1$' by '$\gamma(\alpha)$', which is understood to be $\gamma(\overline{\alpha}N) - 1$, assuming the latter is at least zero, i.e. for large enough $N$. An equality '$\gamma(\alpha) =_0 m_0^0$' is interpreted as $(\forall n^0)\big(\gamma(\overline{\alpha}n) > 0 \to \gamma(\overline{\alpha}n) =_0 m_0 + 1\big)$. In the presence of $\mathsf{QF\text{-}AC}^{1,0}$, $\gamma(\alpha)$ may be defined as $\gamma(\overline{\alpha}F(\alpha)) - 1$, where $F$ originates from the former choice axiom. With these conventions, $\mathsf{NPC}^{\mathfrak{a}}$ (resp. $\mathsf{GH}^{\mathfrak{a}}_{\mathsf{ns}}$) as follows make perfect sense in $\mathsf{P}$ (resp. $\mathsf{P} + \mathsf{QF\text{-}AC}^{1,0}$).

$$(\forall^{\mathrm{st}}\gamma^1 \in C, \alpha^1)(\forall \beta^1)(\alpha \approx_1 \beta \to \gamma(\alpha) =_0 \gamma(\beta)), \tag{$\mathsf{NPC}^{\mathfrak{a}}$}$$

$$\begin{aligned}(\exists^{\mathrm{st}}\Gamma^3)\big[&(\forall^{\mathrm{st}}\gamma^1 \in C)(\forall s^0)\mathsf{GH}(\Gamma, \gamma, s) \\ &\wedge (\forall^{\mathrm{st}}\gamma^1 \in C, s^0)(\forall N \in \Omega)(\hat{H}(\gamma, s, N) = \Gamma(\gamma, s))\big],\end{aligned} \tag{$\mathsf{GH}^{\mathfrak{a}}_{\mathsf{ns}}$}$$

where $\hat{H}$ is $H$ but with $Y(s * 11\dots)$ rather than $Y(s * 00\dots)$ in the first case.

**Theorem 4.21.** *The system* $\mathsf{P}$ *proves* $\mathsf{NPC}^{\mathfrak{a}} \leftrightarrow \Pi^0_1\text{-}\mathsf{TRANS}$*.*

*Proof.* For the implication $\Pi^0_1\text{-}\mathsf{TRANS} \to \mathsf{NPC}^{\mathfrak{a}}$, fix standard $\gamma^1 \in C$, standard $\alpha^1$, and any $\beta^1 \approx \alpha^1$. Now consider $(\exists k^0)\gamma(\overline{\alpha}k) > 0$, implying $(\exists^{\mathrm{st}}k_0^0)\gamma(\overline{\alpha}k) > 0$ by $\Pi^0_1\text{-}\mathsf{TRANS}$. As $\gamma$ is assumed to be a neighbourhood function, we also have $\gamma(\overline{\beta}k_0) > 0$ for the same $k_0$. Since $\alpha \approx_1 \beta$, we obtain $\gamma(\alpha) = \gamma(\beta)$, and $\mathsf{NPC}^{\mathfrak{a}}$.

Working in $\mathsf{P} + \mathsf{NPC}^{\mathfrak{a}}$, suppose $\Pi^0_1\text{-}\mathsf{TRANS}$ is false, i.e. there is standard $h^1$ such that $(\forall^{\mathrm{st}}n)h(n) = 0$ and $(\exists m_0)h(m_0) \neq 0$. Define standard $\gamma^1$ as follows:

$$\gamma(\sigma) := \begin{cases} 1 + \sigma((\mu n \leq |\sigma|)h(n) \neq 0) & (\exists n \leq |\sigma|)h(n) \neq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{4.32}$$

Clearly, $\gamma^1 \in C$ and $\mathsf{NPC}^{\mathbb{c}}$ implies that the latter is nonstandard continuous. However, for $\beta_M := \overline{00\dots M} * MM\dots$, we note that $\beta_0 \approx_1 \beta_{m_0}$ and $\gamma(\beta_0) = 0 \neq m_0 = \gamma(\beta_{m_0})$ if $h(m_0) \neq 0$. This contradiction yields $\mathsf{NPC}^{\mathbb{c}} \to \Pi_1^0\text{-}\mathsf{TRANS}$. □

Let $\mathsf{MU}(\mu)$ be the formula in square brackets in $(\mu^2)$ and let $\mathsf{MPC}^{\mathbb{c}}(\Psi^2)$ be

$$(\forall \gamma^1 \in C)(\forall f^1, g^1 \leq_1 1)(\overline{g}\Psi(\gamma, f) =_0 \overline{f}\Psi(\gamma, f) \to \gamma(f) =_0 \gamma(g)).$$

The innermost formula in $\mathsf{MPC}^{\mathbb{c}}(\Psi)$ is not quantifier-free due to '$\gamma(f) =_0 \gamma(g)$'.

**Corollary 4.22.** *From the proof of* $\mathsf{NPC}^{\mathbb{c}} \leftrightarrow \Pi_1^0\text{-}\mathsf{TRANS}$ *in* $\mathsf{P}$*, terms* $s, t$ *can be extracted such that* $\mathsf{E\text{-}PA}^{\omega*}$ *proves*

$$(\forall \mu^2)\big[\mathsf{MU}(\mu) \to \mathsf{MPC}^{\mathbb{c}}(s(\mu))\big] \wedge (\forall \Psi^2)\big[\mathsf{MPC}^{\mathbb{c}}(\Psi) \to \mathsf{MU}(u(\Psi))\big]. \tag{4.33}$$

*Proof.* To obtain a normal form for $\mathsf{NPC}^{\mathbb{c}}$, proceed in the same way as for $\mathsf{NPC}$ and (4.13). The normal form for $\Pi_1^0\text{-}\mathsf{TRANS}$ is obvious, namely:

$$(\forall^{\mathrm{st}} f^1)(\exists^{\mathrm{st}} n)\big[(\exists m) f(m) \neq 0 \to (\exists i^0 \leq n) f(i) \neq 0\big].$$

Now bring $\mathsf{NPC}^{\mathbb{c}} \leftrightarrow \Pi_1^0\text{-}\mathsf{TRANS}$ in normal form and apply Corollary 2.4. □

In light of Theorem 4.1 and Corollary 4.2, it is easy to obtain a proof of $\Pi_1^0\text{-}\mathsf{TRANS} \to \mathsf{GH}_{\mathsf{ns}}^{\mathbb{c}}$ in $\mathsf{P} + \mathsf{QF}^{1,0} + \mathsf{STP}$. One can show that $\mathsf{GH}_{\mathsf{ns}}^{\mathbb{c}}$ makes senses in $\mathsf{P} + \Pi_1^0\text{-}\mathsf{TRANS}$, i.e. $\mathsf{QF\text{-}AC}^{1,0}$ is not needed. The more interesting reversal is as follows.

**Theorem 4.23.** *The system* $\mathsf{P} + \mathsf{QF\text{-}AC}^{1,0}$ *proves* $\mathsf{GH}_{\mathsf{ns}}^{\mathbb{c}} \to \Pi_1^0\text{-}\mathsf{TRANS}$.

*Proof.* Working in $\mathsf{P} + \mathsf{QF\text{-}AC}^{1,0} + \mathsf{GH}_{\mathsf{ns2}}^{\mathbb{c}}$, we show that every $\gamma_0^1 \in C$ is nonstandard continuous, implying $\mathsf{NPC}^{\mathbb{c}}$ and thus $\Pi_1^0\text{-}\mathsf{TRANS}$ by Theorem 4.21. If $(\forall^{\mathrm{st}} \alpha^1)(\exists^{\mathrm{st}} N)\gamma(\overline{\alpha}N) > 0$ for $\gamma_0^1 \in C$, then the latter is nonstandard continuous. We now derive a contradiction from $(\exists^{\mathrm{st}} \alpha_0^1)(\forall^{\mathrm{st}} N)\gamma_0(\overline{\alpha}N) = 0$ for some fixed $\gamma_0^1 \in C$. Fix such $\alpha_0, \gamma_0$ and define the standard function $\gamma^1 \in C$ as follows:

$$\gamma(\sigma) := \begin{cases} 1 + \sigma((\mu n \leq |\sigma|)\gamma_0(\overline{\alpha_0}n) \neq 0) & (\exists n \leq |\sigma|)\gamma_0(\overline{\alpha_0}) \neq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{4.34}$$

Note that $\gamma(\alpha) = \alpha(m_0)$ where $m_0$ is the least $m$ such that $\gamma_0(\overline{\alpha}m) \neq 0$. Now compute $\Gamma(\gamma, \langle\rangle)$ as follows: $\Gamma(\gamma, \langle\rangle) = \gamma(0 * (\lambda n)\Gamma(\gamma, n+1)) = \Gamma(\gamma, m_0)$ and

$$\Gamma(\gamma, m_0) = \gamma(m_0 * 0 * (\lambda n)\Gamma(\gamma, m_0 * (n+1))) = \Gamma(\gamma, m_0 * (m_0 - 1))$$

Similarly, we have $\Gamma(\gamma, m_0 * (m_0 - 1)) = \Gamma(\gamma, m_0 * (m_0 - 1) * (m_0 - 2))$, and ultimately

$$\Gamma(\gamma, \langle\rangle) = \Gamma(\gamma, m_0 * (m_0 - 1) * \dots * 1) = \gamma(m_0 * (m_0 - 1) * \dots * 1 * 0 * (\lambda n)(\dots)) = 0.$$

However, by the definition of $\hat{H}$, we also have that

$$\hat{H}(\gamma, \langle\rangle, m_0 - 2) = \gamma(0 * \hat{H}(\gamma, 1, m_0 - 2) * \dots \hat{H}(\gamma, m_0 - 1, m_0 - 2) * 11\dots) = 1,$$

where $m_0$ is again the least $m$ such that $h(m) \neq 0$. This contradiction yields $\Pi_1^0\text{-}\mathsf{TRANS}$, and we are done. □

The associated principles and corollary are now as follows.

$$(\forall \gamma^1 \in C, s^0)\big[\Gamma(\gamma, s^0) = \gamma\big(s * 0 * (\lambda n^0)\Gamma(\gamma, s * (n+1))\big)\big]. \tag{$\mathsf{GH}^{\mathbb{c}}(\Gamma)$}$$

$$(\forall \gamma^1 \in C, s)(\forall N \geq \Psi(\gamma, s))\big(\hat{H}(\gamma, s, N) =_0 \Gamma(\gamma, s)\big) \tag{$\mathsf{GHS}^{\mathbb{c}}(\Psi, \Gamma)$}$$

**Corollary 4.24.** *From the proof of* $\mathsf{GH}^{\complement}_{\mathsf{ns}} \to \Pi^0_1\text{-}\mathsf{TRANS}$ *in* $\mathsf{P} + \mathsf{QF\text{-}AC}^{1,0}$*, a term* $t$ *can be extracted such that* $\mathsf{E\text{-}PA}^{\omega*} + \mathsf{QF\text{-}AC}^{1,0}$ *proves that*

$$(\forall \Psi^3, \Gamma^3)\big[[\mathsf{GH}^{\complement}(\Gamma) \wedge \mathsf{GHS}^{\complement}(\Psi, \Gamma)] \to \mathsf{MU}(t(\Psi, \Gamma))\big]. \tag{4.35}$$

*Proof.* Similar to the proof of Corollary 4.22. □

Note that the choice principle $\mathsf{QF\text{-}AC}^{1,0}$ is internal, implying that the term in (4.35) is still a term from Gödel's $\mathsf{T}$, i.e. not involving choice functionals from $\mathsf{QF\text{-}AC}^{1,0}$, by [4, Theorem 7.7]. Furthermore, following the proof of Theorem 4.23 in detail, it becomes clear that the condition '$\Gamma$ is standard' is superfluous, implying that the term in (4.35) really only depends on $\Psi$.

Finally, we study the following variation of $\mathsf{GH}^{\complement}_{\mathsf{ns}}$:

$$\begin{aligned}(\exists^{\mathrm{st}}\Gamma^3)\big[&(\forall^{\mathrm{st}}\gamma^1 \in C)(\forall s^0)\mathsf{GH}(\Gamma, \gamma, s) \\ &\wedge (\forall^{\mathrm{st}}\gamma^1, \varepsilon^1 \in C, s^0)(\gamma \approx_1 \varepsilon \to \Gamma(\gamma, s) =_0 \Gamma(\varepsilon, s)\big],\end{aligned} \tag{$\mathsf{GH}^{\complement}_{\mathsf{ns2}}$}$$

which expresses that the Gandy-Hyland functional is *standard extensional* similar to $(\mathsf{E})^{\mathrm{st}}$ defined in Remark 2.7. Note that $\Pi^0_1\text{-}\mathsf{TRANS}$ immediately yields standard extensionality $(\mathsf{E})^{\mathrm{st}}$ from 'usual' extensionality $(\mathsf{E})$ for standard functionals $\varphi^2$. The reverse implication is again more interesting.

**Theorem 4.25.** *The system* $\mathsf{P} + \mathsf{QF\text{-}AC}^{1,0}$ *proves* $\mathsf{GH}^{\complement}_{\mathsf{ns2}} \to \Pi^0_1\text{-}\mathsf{TRANS}$.

*Proof.* As in the proof of Theorem 4.23, suppose $(\exists^{\mathrm{st}}\alpha^1_0)(\forall^{\mathrm{st}}N)\gamma_0(\overline{\alpha}N) = 0$ for some fixed $\gamma^1_0 \in C$. Let $\gamma^1 \in C$ be as in (4.34) and recall that $\Gamma(\gamma, \langle\rangle) = 0$ by the proof of Theorem 4.23. Now define $\varepsilon^1 \in C$ as in (4.34), but with $2 + \dots$ in the first case, and note that $\gamma \approx_1 \varepsilon$ but $\Gamma(\varepsilon, \langle\rangle) = 1 + \dots \neq 0$. This contradiction yields $\mathsf{NPC}^{\complement}$ and hence $\Pi^0_1\text{-}\mathsf{TRANS}$ by Theorem 4.21. □

Recall the notion of 'extensionality functional' from Section 2.3. Note that an extensionality functional for $\Gamma^3$ from $\mathsf{GH}^{\complement}_{\mathsf{ns2}}$ makes sense in the presence of $\mathsf{QF\text{-}AC}^{1,0}$.

**Corollary 4.26.** *From the proof of* $\mathsf{GH}^{\complement}_{\mathsf{ns2}} \to \Pi^0_1\text{-}\mathsf{TRANS}$ *in* $\mathsf{P} + \mathsf{QF\text{-}AC}^{1,0}$*, a term* $t$ *can be extracted such that* $\mathsf{E\text{-}PA}^{\omega*} + \mathsf{QF\text{-}AC}^{1,0}$ *proves that*

$$(\forall \Xi^2, \Gamma^3)\big[[\mathsf{GH}^{\complement}(\Gamma) \wedge \mathsf{EXT}(\Xi, \Gamma)] \to \mathsf{MU}(t(\Xi, \Gamma))\big]. \tag{4.36}$$

*Proof.* Similar to the proof of Corollary 4.22. The normal form of standard extensionality as in $\mathsf{GH}^{\complement}_{\mathsf{ns}}$ is as follows:

$$(\forall^{\mathrm{st}}\gamma^1, \varepsilon^1 \in C, s^0)(\exists^{\mathrm{st}}N)(\overline{\gamma}N =_0 \overline{\varepsilon}N \to \Gamma(\gamma, s) =_0 \Gamma(\varepsilon, s)\big],$$

which explains the provenance of the extensionality functional in (4.36). □

Note that the existence of an extensionality functional follows trivially from $(\mathsf{E})$ and $\mathsf{QF\text{-}AC}^{1,0}$. In other words, (4.36) expresses $(\mu^2)$ in terms of the $\Gamma$-functional defined on associates, quite similar to [15, Theorem 9.5.4, p. 460]. Of course, the latter theorem is formulated with partial functionals, while all functionals in $\mathsf{P}$ are total. We show in the next section that $\mathsf{P}$ can 'simulate' partiality (relative to the standard world); We also argue that this 'standard partiality' explains the results in this section.

4.7. **Concluding remarks.** We have shown that certain theorems from Nonstandard Analysis give rise to (effective) relative computability results. This resonates nicely with the longstanding (but speculative) claim that Nonstandard Analysis is somehow 'constructive' or 'effective', captured well by the quote:

> It has often been held that nonstandard analysis is highly nonconstructive, thus somewhat suspect, depending as it does upon the ultrapower construction to produce a model [...] On the other hand, nonstandard *praxis* is remarkably constructive; having the extended number set we can proceed with explicit calculations. (Emphasis in original: [1, p. 31])

Similar observations regarding the 'constructive or effective content of Nonstandard Analysis' are made in numerous places; An incomplete list may be found in [20, §1]. The results in this paper can be said to make the aforementioned speculative claim regarding the effective content of Nonstandard Analysis *more concrete*.

By contrast, the following final remark is somewhat vague and speculative, but partially explains the connection between the totality of the Gandy-Hyland functional mentioned in [15, Theorem 9.5.4, p. 460] and Corollaries 4.24 and 4.26.

**Remark 4.27** (Partiality in $\mathsf{P}$)**.** The class of *partial computable functions* is a central object of study in computability theory ([22, I.2.2]). As discussed in the latter, there are good reasons to study partial functions. We now discuss how $\mathsf{P}$ can accommodate partial functionals, *despite all functionals being total.* Intuitively speaking, we show that a total computable function *with standard index* can output nonstandard numbers for standard input (after running for nonstandard many steps). Such a total function may rightly be called 'not total from the point of view of the standard world' in view of the basic axioms of $\mathsf{P}$. More formally:

First of all, consider the well-known predicate '$\varphi^A_{e,s}(n) = m$' which intuitively states that: 'the $e$-th Turing machine with oracle set $A$ and input $n$ halts after $s$ steps with output $m$' ([22, Def. 3.8]). Now let $e_0, x_0$ be *standard* numbers and $A$ a *standard* set such that $(\exists s^0, m^0)[\varphi^A_{e_0,s}(x_0) = m]$, i.e. we say that '$\varphi^A_{e_0}(x_0)$' is defined in the usual computability-theoretic terminology.

Secondly, the basic axioms of $\mathsf{P}$ in Definition 2.1 guarantee that every standard functional evaluated at a standard input returns a standard output. By contrast, *without the presence of* $\Pi^0_1$-$\mathsf{TRANS}$, $\varphi^A_{e_0}(x_0)$ as defined above[9] *may well be nonstandard*. In other words, while the value of $\varphi^A_{e_0}(x_0)$ is defined and all inputs are standard, the $e_0$-th Turing machine may well take a nonstandard number $s$ of steps to halt, with a nonstandard output $m$, as discussed in Footnote 9.

Thirdly, in light of the previous, we are led to the following definition: For standard $e_0, x_0, A$, we say that '$\varphi^A_{e_0}(x_0)$ is *standard-defined*' if $(\exists^{\text{st}} s^0, m^0)[\varphi^A_{e_0,s}(x_0) = m]$, and 'standard-undefined' otherwise. Similarly, for standard $e_0, A$, we say that

[9] Let $h^1$ be the counterexample to $\Pi^0_1$-$\mathsf{TRANS}$ from the proof of Theorem 4.21, and define $e_0$ as the program which tests whether $x_0 \in A := \{n : h(n) \neq 0\}$ and outputs $x_0$ if so, and repeats the previous step for $x_0 + 1$ otherwise.

'$\varphi^A_{e_0}$ is standard-total' if we have $(\forall^{\rm st} x_0)(\exists^{\rm st} s^0, m^0)[\varphi^A_{e_0,s}(x_0) = m]$, and 'standard-partial' otherwise. Hence, define $\psi^A_e$ as follows for $M \in \Omega$:

$$\psi^A_e(x) := \begin{cases} \varphi^A_e(x) & (\exists s^0, m^0)[\varphi^A_{e,s}(x) = m] \\ M & \text{otherwise} \end{cases}$$

Note that this definition makes sense in $\mathsf{P}+(\mu^2)$ and recall that an *internal* sentence such as $\Delta_{\mathsf{int}} \equiv (\mu^2)$ has no impact on the term extraction procedure in Theorem 2.3. In particular, the functional from $\Delta_{\mathsf{int}} \equiv (\mu^2)$ does not occur in the term $t$ from the latter theorem. By definition, $\psi^A_e$ is total *but not standard-total* in the presence of $\neg\Pi^0_1$-$\mathsf{TRANS}$ by Footnote 9. Hence, we can in fact *simulate* the concept of partiality inside $\mathsf{P}$ by exploiting the dichotomy between 'standard' and 'nonstandard'. Similar definitions are possible for higher-type functionals.

Finally, we arrive at the motivation for the above definitions: Consider $\gamma^1 \in C$ as in (4.32); To compute $\gamma(\alpha)$ at standard $\alpha^1$, one evaluates $\gamma(\overline{\alpha}0)$, $\gamma(\overline{\alpha}1)$, et cetera. This computation always terminates by the definition of $\gamma^1 \in C$. However, in the presence of $\neg\Pi^0_1$-$\mathsf{TRANS}$, this computation only terminates after a nonstandard number of steps, i.e. $\gamma^1 \in C$ can be 'standard-partial' in the above sense. However, $\mathsf{GH}^{\mathbb{C}}_{\mathsf{ns}}$ and $\mathsf{GH}^{\mathbb{C}}_{\mathsf{ns2}}$ guarantee that there is a standard-total Gandy-Hyland functional, and by definition of the latter, $\gamma$ evaluates to a standard number for certain (standard) input sequences, i.e. $\gamma$ is also standard-total, and hence nonstandard continuous. Thus follows $\mathsf{NPC}^{\mathbb{C}}$ and hence $\Pi^0_1$-$\mathsf{TRANS}$ by Theorem 4.21.

**Acknowledgement 4.28.** This research was sponsored by the John Templeton Foundation, the FWO Flanders, the University of Oslo, and the Alexander von Humboldt Foundation. The author is grateful to these institutions for their support. The author thanks Dag Normann and Paulo Oliva for their valuable advice.

## References

[1] Sergio Albeverio, Raphael Høegh-Krohn, Jens Erik Fenstad, and Tom Lindstrøm, *Nonstandard methods in stochastic analysis and mathematical physics*, Pure and Applied Mathematics, vol. 122, Academic Press, 1986.

[2] Jeremy Avigad and Solomon Feferman, *Gödel's functional ("Dialectica") interpretation*, Handbook of proof theory, Stud. Logic Found. Math., vol. 137, 1998, pp. 337–405.

[3] Michael J. Beeson, *Foundations of constructive mathematics*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 6, Springer, 1985. Metamathematical studies.

[4] Benno van den Berg, Eyvind Briseid, and Pavol Safarik, *A functional interpretation for nonstandard arithmetic*, Ann. Pure Appl. Logic **163** (2012), no. 12, 1962–1994.

[5] Benno van den Berg and Sam Sanders, *Transfer equals Comprehension*, Submitted (2014). Available on arXiv: `http://arxiv.org/abs/1409.6881`.

[6] Ulrich Berger and Paulo Oliva, *Modified bar recursion*, Math. Structures Comput. Sci. **16** (2006), no. 2, 163–183.

[7] ———, *Modified bar recursion and classical dependent choice*, LC01, Lect. Notes Log., vol. 20, 2005, pp. 89–107.

[8] Martín Escardó, *Seemingly impossible functional programs.* See `http://math.andrej.com/2007/09/28/seemingly-impossible-functional-programs/` and `http://www.cs.bham.ac.uk/~mhe/.talks/popl2012/`.

[9] Martín Escardó, Paulo Oliva, and Thomas Powell, *System T and the product of selection functions*, Computer science logic 2011, LIPIcs. Leibniz Int. Proc. Inform., vol. 12, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2011, pp. 233–247.

[10] Martín Escardo and Chuangjie Xu, *The Inconsistency of a Brouwerian Continuity Principle with the CurryHoward Interpretation*, 13th International Conference on Typed Lambda Calculi and Applications (TLCA 2015), 2015, pp. 153–164.

[11] Yu. L. Ershov, *Model* **C** *of partial continuous functionals*, Logic Colloquium 76 (Oxford, 1976), North-Holland, Amsterdam, 1977, pp. 455–467. Studies in Logic and Foundations in Math., Vol. 87.
[12] Robin Gandy and Martin Hyland, *Computable and recursively countable functions of higher type*, North-Holland, 1977, pp. 407–438. Studies in Logic and Found. Math 87.
[13] Ulrich Kohlenbach, *Higher order reverse mathematics*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, ASL, 2005, pp. 281–295.
[14] ———, *Foundational and mathematical uses of higher types*, Reflections on the foundations of mathematics (Stanford, CA, 1998), Lect. Notes Log., vol. 15, ASL, 2002, pp. 92–116.
[15] John Longley and Dag Normann, *Higher-order Computability*, Theory and Applications of Computability, Springer, 2015.
[16] Edward Nelson, *Internal set theory: a new approach to nonstandard analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 6, 1165–1198.
[17] Dag Normann, *Recursion on the countable functionals*, LNM 811, vol. 811, Springer, 1980.
[18] Abraham Robinson, *Non-standard analysis*, North-Holland, Amsterdam, 1966.
[19] Sam Sanders, *Taming the Reverse Mathematics zoo*, Submitted, Available from arXiv: `http://arxiv.org/abs/1412.2022` (2014).
[20] ———, *The unreasonable effectiveness of Nonstandard Analysis*, Submitted; Available from arXiv: `http://arxiv.org/abs/1508.07434` (2015).
[21] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, CUP, 2009.
[22] Robert I. Soare, *Recursively enumerable sets and degrees*, Perspectives in Mathematical Logic, Springer, 1987.
[23] Anne Sjerp Troelstra, *Metamathematical investigation of intuitionistic arithmetic and analysis*, Springer Berlin, 1973. Lecture Notes in Mathematics, Vol. 344.
[24] Anne Sjerp Troelstra and Dirk van Dalen, *Constructivism in mathematics. Vol. II*, Studies in Logic and the Foundations of Mathematics, vol. 123, North-Holland, 1988.